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SUMMARY

The energy transfer from the uniform flow outside a boundary layer to a transverse surface wave at the boundary is calculated on the hypothesis that the boundary layer may be represented by an inviscid, approximately parallel shear flow. This energy transfer is found to consist of two components; the first is similar to that found previously for supersonic panel flutter in the absence of a boundary layer and is relatively diminished by the presence of the boundary layer; the second is intrinsic to the shear flow and is present whenever the surface wave speed is smaller than the free stream speed (whether subsonic or supersonic). Approximate solutions to the differential equations for the disturbed motion of the boundary layer are es~~tablished~~ and compared with more exact ~~numerical solutions.~~ The application of the results to actual flutter calculations is discussed and an example of a pressurized cylinder considered. It is concluded that the presence of a typical boundary layer may reduce the degree of instability for supersonic flutter of a long, monocoque shell by an order of magnitude, thereby providing a possible explanation of discrepancies between earlier theories (in which boundary layer effects were neglected) and observation. ( )

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CONTENTS

ILLUSTRATIONS

1. INTRODUCTION

The dynamic instability (flutter) of an infinite panel exposed to a supersonic air stream has been discussed in references 1 and 2. It was found there that the flutter speed was approximately $a_1 + (V_o)_{min}$, where a_1 denotes sonic speed and V_o the speed of a transverse wave on the free panel, and that the wavelengths for monocoque structures were sufficiently small to justify the neglect of panel curvature on the aerodynamic (but not the structural) forces and to justify the assumption of an infinite panel. The smallness of the wavelength implies that the effects of a relatively thick boundary layer may be important, and the following analysis is intended to establish the qualitative nature of boundary layer effects and to provide approximations to their magnitude.

We adopt as a model for our calculations an inviscid, parallel shear flow over an infinite, approximately plane panel. Assuming a periodic surface wave displacement of the panel, we will establish the equations of motion for this shear flow and the resulting pressure on the boundary in Section 2, after which the eigenvalue problem for the panel will be established in Section 3. We then will consider, in Section 4, the general nature of the possible instability and show that, in addition to the essentially supersonic instability of reference 2, there also appears a new type of instability associated with energy transfer from the shear flow; this latter mechanism is closely related to that which enters the boundary layer stability problem.

The boundary value problem considered in Section 2 leads to a second-order differential equation that has a regular singularity in the range of integration if the surface wave speed is less than the air speed at the outer (free stream) edge of the boundary layer. We consider, in Section 5, the nature of solutions to this differential equation in the neighborhood of its singularity, its solution by expansion in powers of the ratio of boundary layer thickness to wavelength, and its solution by numerical integration. The application of these results to actual flutter analyses is discussed in Sections 6 and 7.

We emphasize that the model on which the present analysis is based is highly idealized, and experimental confirmation of the results is most desirable; moreover, even though the predicted conditions for the initiation of flutter should prove correct, nonlinear effects must be decisive in determining the severity of such flutter. In the absence of experimental confirmation and at least a qualitative understanding of these nonlinear effects (especially with reference to structural fatigue), our results must be accepted with considerable reserve and should not be regarded as adequate for major design decisions. It appears, nevertheless, that typical boundary layers may reduce the negative damping factor--i.e., the degree of instability--by an order of magnitude (this statement applies only to long, monocoque cylinders, for which the critical wavelength and boundary layer thickness are of the same order of magnitude), thereby providing a possible explanation of previous discrepancies between theory and observation.

2. EQUATIONS OF MOTION—BOUNDARY LAYER

We require the equations of motion that govern the disturbed flow produced in a prescribed boundary layer by the surface wave displacement*

$$\zeta = \zeta\,(x - Vt, y) = \exp\left\{ -ik \left[(x - Vt)\cos\alpha + y\sin\alpha\right]\right\} \tag{2.1}$$

of the boundary from its equilibrium position $z = z_o$; x and y, together with z, are Cartesian coordinates in a reference frame fixed in the panel. t is the time, V the phase velocity with respect to the x-axis, α the angle of propagation (of the wave normal) with respect to the x-axis, and k the wave number-- viz.,

$$k = \frac{2\pi}{\lambda_1}, \tag{2.2}$$

λ_1 being the wavelength of the surface wave. The amplitude of the wave, although for convenience posed as unity in Eq. (2.1), is assumed to be small compared with all other characteristic lengths-- in particular, both the wavelength and the boundary layer thickness. We remark that the assumption with respect to boundary layer thickness might not be satisfied in many cases of practical interest; but, of course, the effect of the boundary layer on panel flutter almost certainly would be unimportant if the amplitude were of the same order as or large compared with the boundary layer thickness.

Our model of the boundary layer is an inviscid, non-heat-conducting, parallel, shear flow that, in the absence of the traveling wave disturbance, is characterized by the velocity profile U(z), the temperature profile T(z), and the free stream pressure p_1 (which is constant through the boundary layer). The sonic velocity a(z) and density $\rho(z)$ then are related to the temperature according to

$$\frac{a^2(z)}{a_1^2} = \frac{\rho_1}{\rho(z)} = \frac{T(z)}{T_1} \tag{2.3}$$

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* Cf. Eq. (3.13), Ref. 2, x_1 and y_1 therein being replaced here by x and y.

by virtue of the constant pressure; the subscript 1 on the fluid variables denotes the free stream conditions outside the boundary layer. We neglect the x and y variations of all of these quantities with the implicit assumption that local values may be used (as in the boundary layer stability problem).

If the boundary layer is laminar the essential approximation in our shear flow model is that the viscous forces associated with the traveling wave are negligible compared with the corresponding inertia forces (the neglect of heat conduction is almost certainly of less importance than the neglect of viscosity). This approximation is similar to that adopted in the inviscid approach to the boundary layer stability problem,[3] although in the present problem the approximation remains valid in the neighborhood of the boundary by virtue of the finite acceleration there. As in the boundary layer problem, the approximation breaks down in an <u>inner viscous layer</u> (Ref. 3, page 136), where the inertia forces tend to zero. The result is a singularity in the inviscid equations of motion (see Sections 4 and 5), which may be said to represent the inner viscous layer in the limit of zero viscosity. We infer from the known results for the boundary layer stability problem that this representation is significant for small but finite values of the viscosity if the perturbation motion is unstable.

If the boundary layer is turbulent and we interpret U, T, ρ, a, and p_1 as mean values, the approximations implied by the model are more severe. First, we neglect the perturbation Reynolds stresses associated with first order* coupling between the perturbation flow and fluctuations in the original flow. Secondly, we implicitly neglect cross-correlations between the fluctuations of unlike quantities -- *e.g.*, density and velocity. We suggest that the latter approximation is not likely to introduce uncertainties greater than those inherent in presently available profile data for compressible boundary layers; this conjecture is supported by the observation that these cross-correlations would not appear in the corresponding incompressible problem and probably represent only minor compressibility effects compared with the variation of mean Mach number (which <u>is</u> incorporated in our model).

We next remark that perturbations having the phase velocity V with respect to the steady shear flow will appear as steady disturbances to an observer moving with this same velocity V, with respect to whom the original flow will have the velocity distribution $U(z)-V$. This allows us to

* Cf. Appendix, Ref. 7.

use Ward's result[4] that the perturbation pressure $p-p_1$ and the perturbation velocity components v and w (along y and z, respectively) may be derived from a potential ϕ according to

$$p_1 - p = \phi_x, \tag{2.4a}$$

$$\rho\,(U - V)\,v = \phi_y, \tag{2.4b}$$

$$\rho\,(U - V)\,w = \phi_z, \tag{2.4c}$$

where ϕ satisfies

$$(1 - M^2)\,\phi_{xx} + \phi_{yy} + \phi_{zz} - \left(\frac{2}{M}\right)(M_y\,\phi_y + M_z\,\phi_z) = 0, \tag{2.5}$$

with M as the apparent (to an observer moving with the wave at elevation z) Mach number

$$M = M(z) = \frac{U(z) - V}{a(z)}. \tag{2.6}$$

We emphasize that ϕ is not a velocity potential in the classical sense and that the flow is not irrotational.

The linearized boundary condition to be imposed on ϕ at the wall ($z = z_o$) is, from Eqs. (2.1) and (2.4c),*

* We note that if $M(z)$ is introduced in place of z as an independent variable the boundary condition (2.7) may be regarded as imposed along the streamline on which $M = M_o = M(z_o)$ in the absence of the disturbance, after which the variable z may be interpreted as a parametric streamline coordinate, rather than the linear distance from the wall. [The validity of this interpretation may be established in two dimensions, say x and z, by introducing the independent variables x and ψ, where ψ is a stream function for the compressible flow, in place of x and z (von Mises transformation) prior to the linearization of the equations of motion.] This strategem avoids the restriction (that would be implied if z were interpreted directly as a linear measure) to amplitudes that are small compared with any distance in the boundary layer over which appreciable changes in velocity occur; such a restriction would be highly undesirable because of the large velocity gradient near the wall. We also note that, while $U = 0$ at the wall, it might be desirable in some applications to impose the boundary condition (2.7) at some other streamline--e.g., the edge of the laminar sublayer; this would be permissible insofar as the distance between this streamline and the wall were negligible compared with the wave-length. In effect, we assume that the sublayer moves with the boundary.

$$\phi_z = \rho(U-V)\frac{D\zeta}{Dt} = -ik\cos\alpha\ \rho(U-V)^2\zeta, \qquad z = z_o. \tag{2.7}$$

This boundary condition (together with the anticipated form of the expression for the pressure at $z = z_o$) suggests that we assume a solution to Eq. (2.5) in the form

$$\phi(x-Vt, y, z) = i\ \rho_1 a_1^2 \sec\alpha\, f(z)\, \zeta\,(x-Vt,\ y)\ . \tag{2.8}$$

Substituting this in Eqs. (2.5) and (2.7) and introducing

$$m = M(z)\cos\alpha = \frac{U(z) - V}{a(z)\sec\alpha}\ , \tag{2.9}$$

we obtain

$$m^2 \frac{d}{dz}\left(m^{-2}\frac{df}{dz}\right) - k^2(1-m^2)f = 0 \tag{2.10}$$

and

$$\left.\frac{df}{dz}\right|_{z=z_o} = -km^2(z_o)\ . \tag{2.11}$$

The differential equation satisfied by $f(z)$, to which the perturbation pressure is proportional, was obtained by Lighthill[5] in his studies on shock wave, boundary layer interaction; it is appreciably simpler than the differential equation satisfied by w (the latter usually has been used in boundary layer stability studies[3]). We also note that if $m = 0$ ($U_1 = V$) at, say, $z = z_c$ in the boundary layer, the differential equation (2.10) has a regular singularity there.* The immediate (z) neighborhood of this singularity constitutes the <u>inner viscous layer</u>, where the inertial forces, being proportional to $(U-V)^2$, cannot dominate the viscous forces; however, the thickness of this layer approaches zero for large Reynolds numbers (Ref. 3, page 136) if the disturbance is self-excited, and we therefore neglect it in the following development (except insofar as it is represented by the singularity at $z = z_c$).

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* This singularity is only apparent if $(U'/T)'$ vanishes at $z = z_c$; cf. Eq. (4.7).

The differential equation satisified by $f(z)$ outside of the boundary layer, where m assumes the constant value m_1, reduces to

$$\frac{d^2f}{dz^2} - \beta^2 f = 0, \qquad (2.12)$$

where

$$\beta = k(1 - m_1^2)^{1/2} = k\left\{ 1 - \left[\frac{U_1 - V}{a_1 \sec \alpha} \right]^2 \right\}^{1/2}. \qquad (2.13)^*$$

We then have

$$f(z) \sim e^{-\beta z}, \qquad z > z_1, \qquad (2.14)$$

where $z = z_1$ denotes the edge of the boundary layer; from this we deduce the boundary condition

$$\frac{df}{dz} + \beta f = 0, \qquad z = z_1. \qquad (2.15)$$

The finiteness condition that f be bounded as $z \rightarrow \infty$ requires $R\ell\,\beta \geq 0$, while the radiation condition requires $\operatorname{Im}\beta$ to have the same sign as $V - U$ if $R\ell\,\beta = 0$. These requirements will be satisfied everywhere in the complex V-plane if we draw the cuts for β from the branch points $V = U_1 \pm a_1 \sec \alpha$ to $\pm\infty$ and exclude points on their top sides. Of course, it is not necessary to satisfy the finiteness and radiation conditions everywhere in the V-plane but only at those points for which solutions to Eq. (2.12) are to be considered (thus, in reference 1, the cuts were drawn into $\operatorname{Im} V > 0$ and only points in $\operatorname{Im} V \leq 0$ considered); in the final analysis, we may deform the cuts in any way that ensures the existence of the solution to the differential equation and the satisfaction of the boundary conditions. We remark, in this connection, that a periodic disturbance of the type (2.1) cannot exist by itself, and in a complete initial value problem we would be led to consider an integral superposition of such disturbances over a prescribed path in the V-plane.

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*In Section 5, β is redefined as $\kappa\,(1 - m_1^2)^{1/2}$, where $\kappa = k\delta$, and δ denotes boundary layer thickness.

The pressure on the panel at $z = z_o$, as obtained from Eqs. (2.4) and (2.8), is given by

$$p_a = (p - p_1)_{z=z_o} = -\rho_1 a_1^2 k F\left(\frac{U_1}{A}, \frac{V}{U_1}\right) \zeta(x - Vt, y), \qquad (2.16)$$

where we have introduced

$$F\left(\frac{U_1}{A}, \frac{V}{U_1}\right) = f(z_o) \qquad (2.17)$$

and

$$A = a_1 \sec \alpha . \qquad (2.18)$$

Our boundary value problem now may be posed as: find a solution to the differential equation (2.10), subject to the boundary conditions (2.11) and (2.15), and evaluate it at $z = z_o$.

3. EQUATIONS OF MOTION – PANEL

We present in this section a brief derivation of the eigenvalue equation for the panel. A more detailed derivation has been given in reference 2.

We assume the existence of an operator L, such that $L\zeta$ is the reaction force per unit area associated with a transverse displacement ζ. L may imply both differential and integral operations with respect to the space coordinates, but not with respect to the time--i.e., $L\zeta$ is in phase with ζ. The equation governing the panel motion then reads

$$L\zeta + \sigma \frac{\partial^2 \zeta}{\partial t^2} = p_i - p_a, \tag{3.1}$$

where p_i denotes the perturbation pressure of the fluid beneath (inside) the panel, p_a the perturbation pressure outside the panel, and σ the panel mass per unit area. If the compressibility of the internal fluid is neglected, p_i represents a virtual inertia force, which, for the traveling wave defined by Eqs. (2.1) and (2.2), is given by

$$p_i = \rho_i k (V \cos \alpha)^2 \zeta, \tag{3.2}$$

where ρ_i is the fluid density. (This follows directly from the fact that the effective depth of an incompressible fluid for a surface wave motion is $1/k$; see reference 2 for an alternative derivation.)

The eigenvalue equation corresponding to the traveling wave of Eq. (2.1) may be derived (see reference 2 for details) by relating the operator L to the wave speed V_o that would result in the absence of the aerodynamic force (but including the virtual inertia force of the internal fluid); thus, substituting ζ and p_i from Eqs. (2.1) and (3.2) in Eq. (3.1) with $V = V_o$ therein, we obtain

$$L\zeta = \left(\sigma + \frac{\rho_i}{k}\right)(k V_o \cos \alpha)^2 \zeta. \tag{3.3}$$

We emphasize that V_o is an eigenvalue of Eq. (3.3) and is a function of k and α.

Substituting p_a from Eq. (2.16), and $L\zeta$ from Eq. (3.3) in the complete equation (3.1), with $V \neq V_o$ in ζ and p_1, and dividing through by the coefficient of V^2, we obtain the eigenvalue equation (for V)

$$V^2 + \mu A^2 F\left(\frac{U_1}{A}, \frac{V}{U_1}\right) = V_o^2 , \tag{3.4}$$

where

$$\mu = \frac{\rho_1}{\rho_1 + \sigma k} \tag{3.5}$$

is the aerodynamic-mass parameter, and F and A are defined by Eqs. (2.17) and (2.18).

4. SOME GENERAL CONSIDERATIONS REGARDING STABILITY

The explicit determination of the stability boundaries for a given panel, as represented in the specification [or derivation from Eq. (3.3)] of V_o as a function of k and α and the mass parameter μ, and a given boundary layer, as represented by the specification of velocity and temperature profiles, requires the simultaneous solution of the differential equation (2.10), subject to the boundary conditions (2.11) and (2.15), and the transcendental equation (3.4).* This represents a formidable problem, which is especially complicated if $V = U$ within the boundary layer, giving rise to a singularity of the differential equation (2.10). In order to clarify the role of this singularity with respect to panel flutter, we develop in this section an implicit solution to the boundary value problem.

We first rewrite Eq. (3.4) in the form

$$V = V_o \left\{ 1 - \mu \left(\frac{A}{V_o} \right)^2 F\left(\frac{U_1}{A}, \frac{V}{U_1} \right) \right\}^{1/2} , \tag{4.1}$$

where the phase of the radical is defined such that $V = V_o$ at $\mu = 0$. It then follows, neglecting structural damping (V_o real), that the imaginary part of V will be opposite in sign to the imaginary part of F, so that instability requires $\operatorname{Im} F > 0$.

We obtain an implicit solution for F by multiplying Eq. (2.10) through by $m^{-2}\bar{f}$ (where the overbar denotes the complex conjugate), integrating from $z = z_o$ to $z = z_1$, integrating the term $\bar{f}\,(m^{-2} f')'$ by parts, imposing the boundary conditions (2.11) and (2.15) to eliminate f' at the limits, and solving for $\bar{f}\,(z_o)$; the end result is

$$F = \bar{f}(z_o) = \left| f(z_1) \right|^2 m_1^{-2} \left(1 - m_1^2 \right)^{1/2} + \frac{1}{k} \int_{z_o}^{z_1} \left[\left| f'(z) \right|^2 + k^2 (1 - m^2) \left| f \right|^2 \right] \frac{dz}{m^2} . \tag{4.2}$$

---

* Alternatively, Eq. (3.4) may be combined with Eq. (2.11) to obtain a homogeneous boundary condition on f at $z = z_o$. This formation leads to a more symmetric boundary value problem but is less satisfactory for the approximate solutions treated herein.

We will assume that the phase angle of V is small (V approximately real); then m will be approximately real, and the imaginary parts of the first and second terms on the right-hand side of Eq. (4.2) must be derived chiefly from the radical $(1 - m_1^2)^{1/2}$ and the singularity at $m = 0$, respectively.

The phase of $(1 - m_1^2)^{1/2} = \beta/k$ for V approximately real is, by definition, 0 for $|m_1| < 1$ or $\pm \pi/2$ for $\pm m_1 > 1$, corresponding to our previous choice of branch cuts for β [see remarks following Eq. (2.15)]. Then, writing

$$F_a = \left|f(s_1)\right|^2 m_1^{-2} (m_1^2 - 1)^{1/2}, \qquad |m_1| > 1 \tag{4.3}$$

we infer that the contribution to $\operatorname{Im} F$ ($= -\operatorname{Im} \bar{F}$) of the first term in Eq. (4.2) is F_a, 0, or $-F_a$ for $V < U_1 - A$, $U_1 - A < V < U_1 + A$, or $V > U_1 + A$, respectively, if V lies close to the real axis but not too close to $U_1 \pm A$. The integral in Eq. (4.2) will contribute an imaginary part, say F_b, to F only if $0 < V < U_1$, in which case m will vanish at the point $s = s_c$. It follows that

$$\begin{aligned} \operatorname{Im} F &\doteq F_a + F_b, && 0 < V < U_1 - A \\ &\doteq F_b, && U_1 - A < V < U_1 \\ &\doteq 0, && U_1 < V < U_1 + A \\ &\doteq -F_a, && U_1 + A < V. \end{aligned} \tag{4.4}$$

We remark that if V does approximate $U_1 \pm A$, as in Section 6, the contribution of the first term in Eq. (4.2) to $\operatorname{Im} F$ must be modified, but this does not affect the contribution of the second term, which remains as F_b or 0.

The nature of the negative damping represented by F_a is essentially the same as that predicted in the absence of the boundary layer and discussed in more detail in reference 2. The presence of the boundary layer reduces this component by partially isolating the surface wave on the panel from the

direct action of the high-speed airflow outside the boundary layer (in particular, the panel is exposed directly only to a subsonic flow, even though the free-stream flow is supersonic).

We may deduce a simplified expression for F_b by noting from the differential equation (2.10) that $f'(z) = O(m)$ and $f(z) = f(z_c) + O(m^2)$ as $z \to z_c$;* accordingly,

$$F_b = -k^{-1} \operatorname{Im} \int_{z_o}^{z_1} \left[|f'|^2 + k^2 (1 - m^2) |f|^2 \right] m^{-2} \, dz \tag{4.5a}$$

$$= -k \, \delta \, |f(z_c)|^2 \operatorname{Im} K_1, \tag{4.5b}$$

where K_1, as defined by

$$K_1 = \delta^{-1} \int_{z_o}^{z_1} (m^{-2} - 1) \, dz, \tag{4.6}$$

is the integral appearing in the solution for $f_1(z)$ discussed in Appendix A. The sign of $\operatorname{Im} K_1$ evidently depends on how the singularity is circumvented; this is an extremely delicate question, but it has been examined thoroughly by Lin (Ref. 3, ch. 8), and we merely state his result that the path of integration must be indented over the point $z = z_c$,** whence

$$F_b = -\pi k a_c^2 \sec^2 \alpha \, |f(z_c)|^2 \left(\frac{T_c}{U_c'^3} \right) \frac{d}{dz} \left(\frac{U'}{T} \right) \bigg|_{z=z_c}, \tag{4.7}$$

where the subscript c implies evaluation at $z = z_c$ (see Appendix B for details).

The integral K_1 occurred in the laminar boundary layer stability problem, and Lees and Lin[6] have shown that if the original flow absorbs energy from a small disturbance in the boundary layer profile

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* Cf. Eq. (5.4b).

** We note that the sign of the wave number usually assumed in boundary layer studies is opposite that chosen here, which has the effect of interchanging i and -i; thus, the path of integration elected by Lin actually is indented under the singular point $z = z_c$.

$$\frac{d}{dz}\left(\frac{U'}{T}\right)\Bigg|_{z=z_c} < 0. \tag{4.8}$$

It does not follow that Eq. (4.8) is a necessary and sufficient condition for the stability of the boundary layer, but it seems likely [on the basis of empirical considerations, as well as Eq. (4.8)] that it will be satisfied for both laminar and turbulent boundary layers in those aerodynamic regimes for which panel flutter presents a serious problem. We may anticipate, therefore, that F_b will be positive, and we infer from this consideration that the boundary layer could be destabilizing with respect to surface waves having wave speeds in the interval $0 < V < U_1$ and amplitudes small compared with both boundary layer thickness and wave-length; however, the net effect in the interval $0 < V < U_1 - A$ is more likely to be stabilizing in consequence of the reduction of F_a.* The destabilizing mechanism represented by F_b appears to be closely related to that studied in the boundary layer stability problem (see reference 3, Section 4.4; also reference 7).

A rough approximation to the value of z_c for a turbulent boundary layer may be deduced from the well-known 1/7 power law--viz.,

$$\frac{U(z)}{U_1} = \left[\frac{z - z_o}{\delta}\right]^{1/7}, \tag{4.9}$$

where δ is the boundary layer thickness. We anticipate that panel flutter will be most serious when V approximates or is less than $U_1 - a_1$ (as in the absence of a boundary layer: cf. reference 2 and Sections 6 and 7 below). Setting $U = U_1 - a_1$ and $z = z_c$ in Eq. (4.9) then yields

$$\frac{z_c - z_o}{\delta} = \left(1 - \frac{1}{M_1}\right)^7, \tag{4.10}$$

---

*We note that $F_b \equiv 0$ in subsonic flow, and instability then would have to be charged entirely to the shear flow in the boundary layer. It seems likely that F_b would be too small to render practical aircraft structures unstable, but it may afford an explanation of the generation of gravity waves on a liquid surface.[7]

which has the values $5 \cdot 10^{-4}$ and $8 \cdot 10^{-3}$ for $M_1 = 1.5$ and 2, respectively; these values are so small that they might place z_c in the laminar sublayer, where the 1/7 power law no longer would be valid, but they may be assumed to give the correct order of magnitude. Of course, $(z_c - z_o)/\delta$, as given by Eq. (4.10), does not remain small for large M_1, but it seems likely that F_b then would be unimportant compared with F_a.

We add that $(z_c - z_o)/\delta$ probably would approximate $(1 - 1/M_1)$ for a laminar boundary layer, but a laminar layer almost certainly would be so thin as to have no effect on panel flutter.

5. THE BOUNDARY VALUE PROBLEM

We now return to the boundary value problem posed by the differential equation (2.10) and the associated boundary conditions (2.11) and (2.15). Introducing the dimensionless variable

$$\zeta = \frac{z - z_0}{\delta} \tag{5.1a}$$

and the boundary layer thickness parameter

$$\kappa = k\delta = 2\pi \frac{\delta}{\lambda_1}, \tag{5.1b}$$

where

$$\delta = z_1 - z_0, \tag{5.1c}$$

in Eqs. (2.10), (2.11), and (2.15), we obtain

$$m^2 \left(m^{-2} f'\right)' - \kappa^2 (1 - m^2) f = 0, \qquad 0 \leq \zeta \leq 1, \tag{5.2}$$

$$f_0' = -\kappa m_0^2, \tag{5.3a}$$

and

$$f_1' + \beta f_1 = 0, \tag{5.3b}$$

where the primes imply differentiation with respect to ζ, the subscripts 0 and 1 imply evaluation at $\zeta = 0$ and 1, and β is defined by Eq. (2.13) if k is replaced by κ therein.

The differential equation (5.2) has a regular singularity at $\zeta = \zeta_c$, where $m(\zeta_c) = 0$. The exponents of this singularity are 3 and 0, and we find that the corresponding solutions in its neighborhood are of the form

$$w_1(\zeta) = (\zeta - \zeta_c)^3 - \frac{3q}{4}(\zeta - \zeta_c)^4 + O(\zeta - \zeta_c)^5 \tag{5.4a}$$

and

$$w_2(\zeta) = \frac{q\kappa^2}{3} w_1(\zeta)\log\left[(\zeta - \zeta_c)e^{-i\pi}\right] + 1 - \frac{\kappa^2}{2}(\zeta - \zeta_c)^2 + O(\zeta - \zeta_c)^4, \tag{5.4b}$$

where

$$q = \frac{-m_c''}{m_c'} . \tag{5.5}$$

The solution w_1 is everywhere real for real values of $\zeta - \zeta_c$, while w_2 has been normalized to have the imaginary part 0 or $(-\pi i q \kappa^2/3)w_1$ as $\zeta < \zeta_c$ or $\zeta > \zeta_c$, respectively (since the logarithmic branch point must be circumvented by indenting over $\zeta = \zeta_c$). The Wronskian of this pair of solutions is

$$W\left\{w_1, w_2\right\} = w_1 w_2' - w_2 w_1' = -\frac{3m^2(\zeta)}{m_c'^2} . \tag{5.6}$$

We also find it convenient to introduce a second pair of solutions, f_1 and f_2, that satisfy the boundary conditions

$$f_{10} = 1, \tag{5.7a}$$

$$f_{10}' = 0, \tag{5.7b}$$

$$f_{20} = 0, \tag{5.7c}$$

and

$$f_{20}' = m_o^2, \tag{5.7d}$$

where the first and second subscripts identify the solution and the point of evaluation, respectively. The corresponding Wronskian is

$$W\left\{f_1, f_2\right\} = m^2(\zeta). \tag{5.8}$$

Invoking the boundary conditions (5.7) on linear combinations of w_1 and w_2 and making use of the Wronskian (5.6), we obtain

$$f_1(\zeta) = \frac{m_c'^2}{3m_o^2}\left[w_{10}' \; w_2(\zeta) - w_{20}' \; w_1(\zeta)\right] \tag{5.9a}$$

and

$$f_2(\zeta) = \frac{m_c'^2}{3}\left[w_{20} \; w_1(\zeta) - w_{10} \; w_2(\zeta)\right]. \tag{5.9b}$$

The required solution, subject to the boundary conditions (5.3), now may be expressed in terms of either w_1 and w_2 or f_1 and f_2. We find

$$f = -\kappa m_o^2\left[\frac{(w_{11}' + \beta w_{11}) \; w_2(\zeta) - (w_{21}' + \beta w_{21}) \, w_1(\zeta)}{(w_{11}' + \beta w_{11}) \; w_{20}' - (w_{21}' + \beta w_{21}) \; w_{10}'}\right] \tag{5.10a}$$

and

$$f = F f_1(\zeta) - \kappa f_2(\zeta), \tag{5.10b}$$

where

$$F = \kappa \frac{f_{21}' + \beta f_{21}}{f_{11}' + \beta f_{11}}. \tag{5.11}$$

We require only the imaginary part of F throughout much of the subsequent analysis. This can be calculated directly from Eq. (5.11), but it may be advantageous to consider separately the components F_a and F_b, as defined by Eqs. (4.3) and (4.5b); these depend only on the absolute value of f and do not require the calculation of phase (which might be rather small in some applications). Setting $\zeta = 1$ in Eq. (5.10b), simplifying the result with the aid of the Wronskian (5.8) evaluated at $\zeta = 1$, and substituting in Eq. (4.3), we obtain

$$F_a = \frac{\kappa^2 m_1^2 (m_1^2 - 1)^{1/2}}{\left|f_{11}' + \beta f_{11}\right|^2}, \qquad m_1 > 1. \tag{5.12}$$

Setting $\zeta = \zeta_c$ in Eq. (5.10b), expressing the denominator of the result in terms of f_1 with the aid of Eq. (5.9a), and substituting in Eq. (4.5b), we obtain

$$F_b = -K^3 m_c'^4 \left[\frac{\left|w_{11}' + \beta w_{11}\right|^2}{9\left|f_{11}' + \beta f_{11}\right|^2}\right] \operatorname{Im} K_1 . \tag{5.13}$$

The solutions f_1, f_2, and w_1 are developed as power series in K in Appendix A. The first approximations to the quantities required for the evaluation of F_a and F_b are

$$f_{11} = 1 + O(K^2), \quad f_{11}' = K^2 m_1^2 K_1 \left[1 + O(K^2)\right], \tag{5.14a,b}$$

$$w_{11} = \frac{3J_1}{m_c'^2}\left[1 + O(K^2)\right], \quad w_{11}' = \frac{3m_1^2}{m_c'^2}\left[1 + O(K^2)\right], \tag{5.15a,b}$$

where K_1 is the integral introduced in Eq. (4.6) and evaluated in Appendix B, and J_1 is defined by Eq. (A15) in Appendix A. Substituting these approximations in Eqs. (5.12) and (5.13) yields

$$F_a = \frac{m_1^2\,(m_1^2 - 1)^{1/2}}{\left|-i(m_1^2 - 1)^{1/2} + K m_1^2 K_1\right|^2}\left[1 + O(K^2)\right], \qquad m_1 > 1, \tag{5.16a}$$

$$= 0, \qquad m_1 < 1, \tag{5.16b}$$

and

$$F_b = -K\left|\frac{m_1^2 + K(1 - m_1^2)^{1/2} J_1}{(1 - m_1^2)^{1/2} + K m_1^2 K_1}\right|^2 \operatorname{Im} K_1 \left[1 + O(K^2)\right] \tag{5.17a}$$

$$= \frac{-K m_1^4 \operatorname{Im} K_1}{\left|(1 - m_1^2)^{1/2} + K m_1^2 K_1\right|^2}\left[1 + O\left(K^2, K\sqrt{1 - m_1^2}\right)\right] \tag{5.17b}$$

The approximation (5.17b) anticipates (on the basis of the results for $K = 0$; see reference 2) that supersonic panel flutter should prove most serious for values of m_1 only slightly larger than unity for sufficiently small values of K.

The approximations to F_a and F_b provided by Eqs. (5.16) and (5.17b), together with the values of K_1 given by Figs. 1a and 1b (for $\alpha = 0$ and an insulated boundary) are plotted in Figs. 2a through 2c and Figures 3a through 3d, for $M_1 = 0.1, 1.2, 1.5,$ and 2.0 and $K = 0.5, 1.0,$ and 2.0. Also plotted (labelled "exact") are some results that were obtained by numerical integration of the differential equation (5.2) on a digital computer.* The agreement doubtless could be improved by including terms of higher order in K in the approximate results for F_a and F_b or, in some cases, merely by using Eq. (5.17a) in place of Eq. (5.17b), but the errors in the approximations (5.16a) and (5.17b) are not likely to exceed those already present in our simplified model of the boundary layer.

We emphasize that the results (both "exact" and "approximate") for F_b are not valid for V/U_1 less than about 0.3 in consequence of the breakdown of the one-seventh power law assumed in the calculations. An adequate correction factor for F_b could be achieved by forming the ratio of $\operatorname{Im} K_1$ for the actual $U(z)$ and $T(z)$ to that based on the one-seventh power law, using Eq. (B8c). The results for F_a do not appear to be very sensitive to changes in the profile.

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*The procedure was to obtain first the regular solution w_1 by integrating away from the singularity, starting the solution by power series expansions near $\zeta = \zeta_c$ and then continuing by the Rungé-Kutta method over the range $(2^{-14}, 1)$. The singular solution w_2 then was expressed as

$$w_2 = (qK^2/3)\, w_1 \log\left[(\zeta - \zeta_c)\, e^{-i\pi}\right] + g(\zeta),$$

the inhomogeneous differential equation for g determined by substitution in the original differential equation, and the regular function g obtained by the procedure outlined for w_1. In those cases where ζ_c was outside of the range of integration, f_1 was determined directly by integrating out from $\zeta = 2^{-14}$. The details and programming were worked out by Dr. Samuel Conte and Mr. David Bussard; the computer was a Remington-Rand, Univac Scientific Model 1103A.

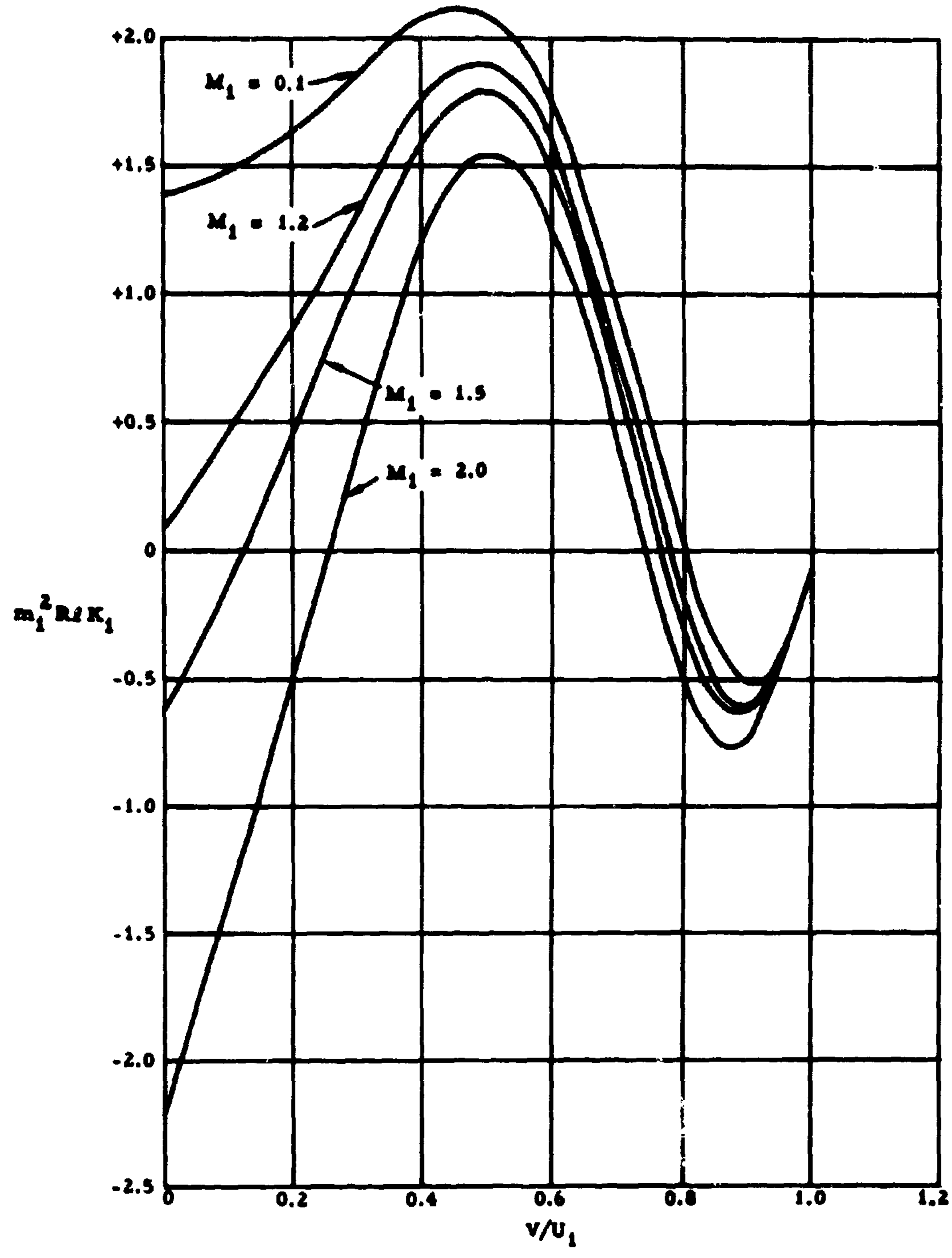

Figure 1a. The real part of $m_1^2 K_1$, as given by Equations (B16) - (B20) for $a = 0$ and an insulated boundary layer; if $a \neq 0$ the results are approximately correct if the nominal value of M_1 on the curves is replaced by $M_1 \cos a$.

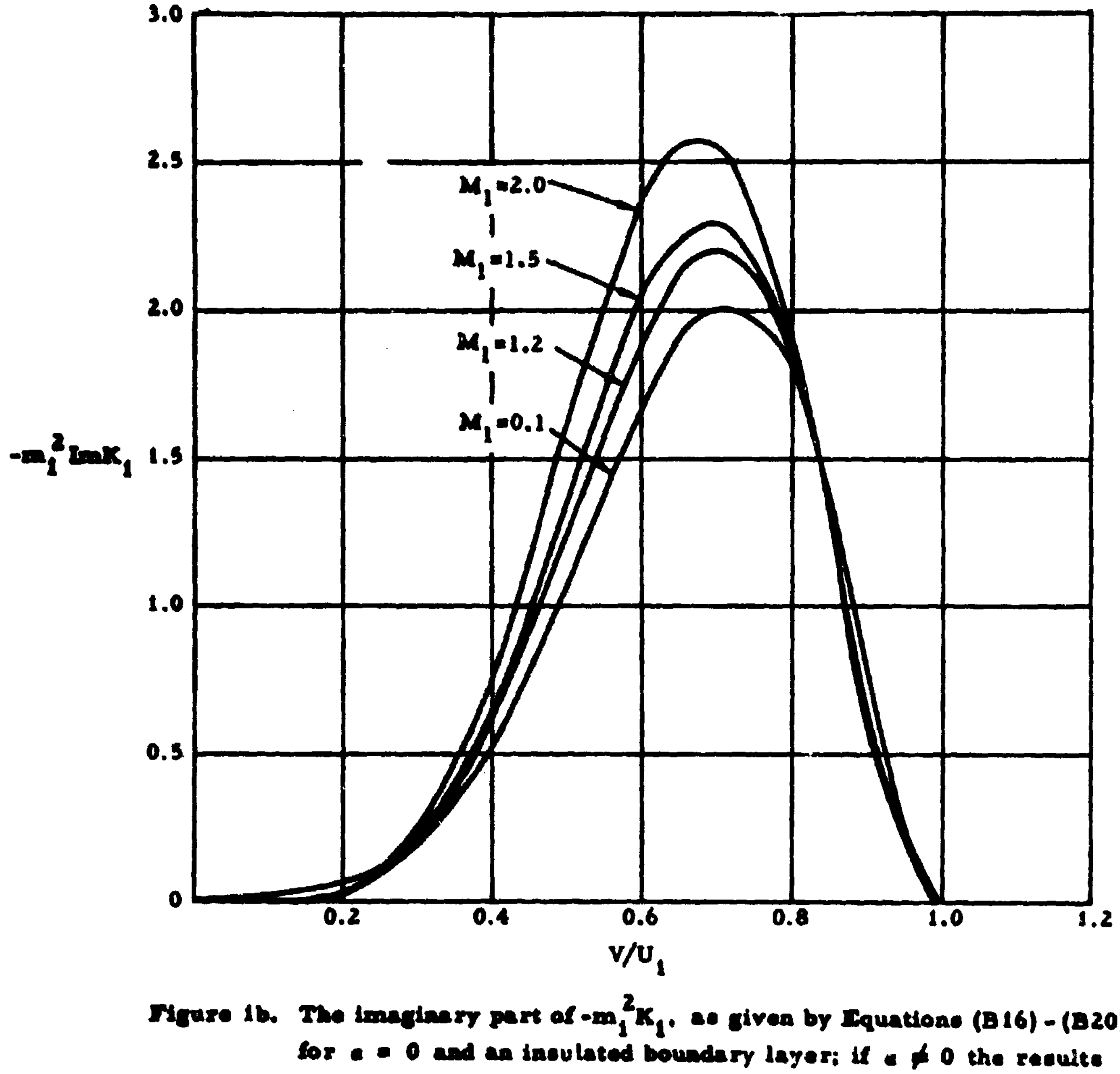

Figure 1b. The imaginary part of $-m_1^2 K_1$, as given by Equations (B16) - (B20) for $\alpha = 0$ and an insulated boundary layer; if $\alpha \neq 0$ the results are approximately correct if the nominal value of M_1 on the curves is replaced by $M_1 \cos \alpha$.

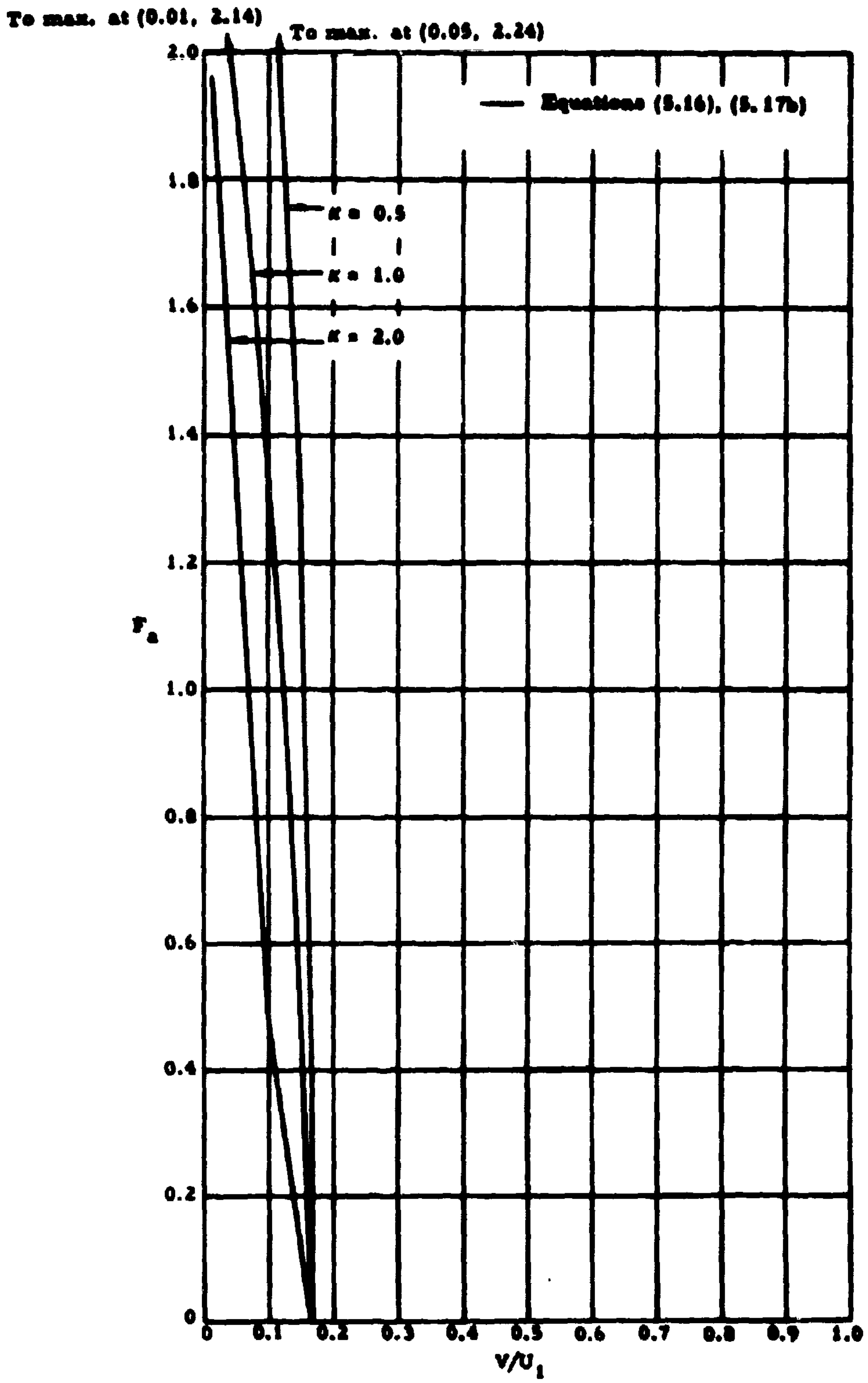

Figure 2a. F_a versus V/U_1 for M_1 = 1.2.

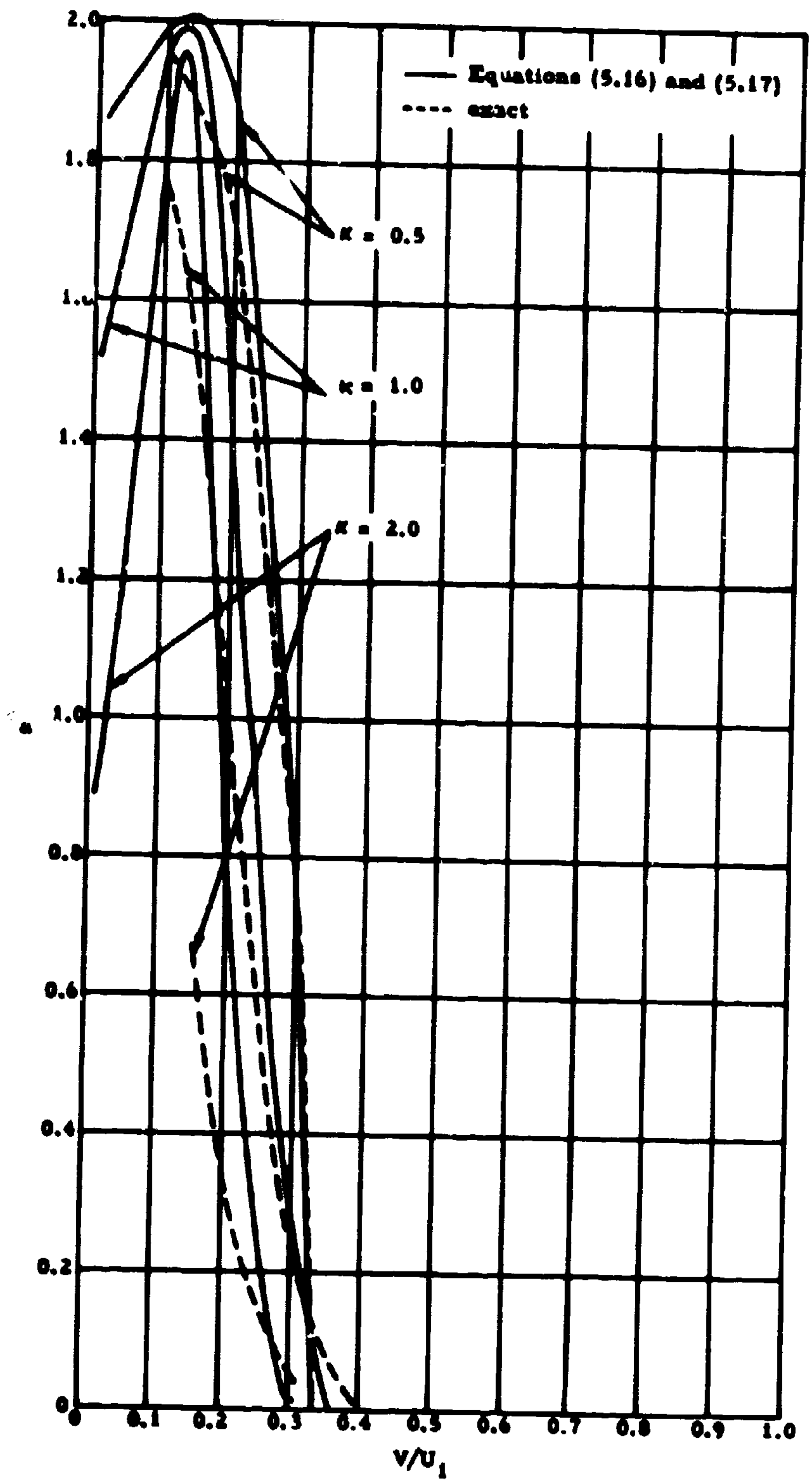

Figure 2b. F_a versus V/U_1 for $M_1 = 1.5$.

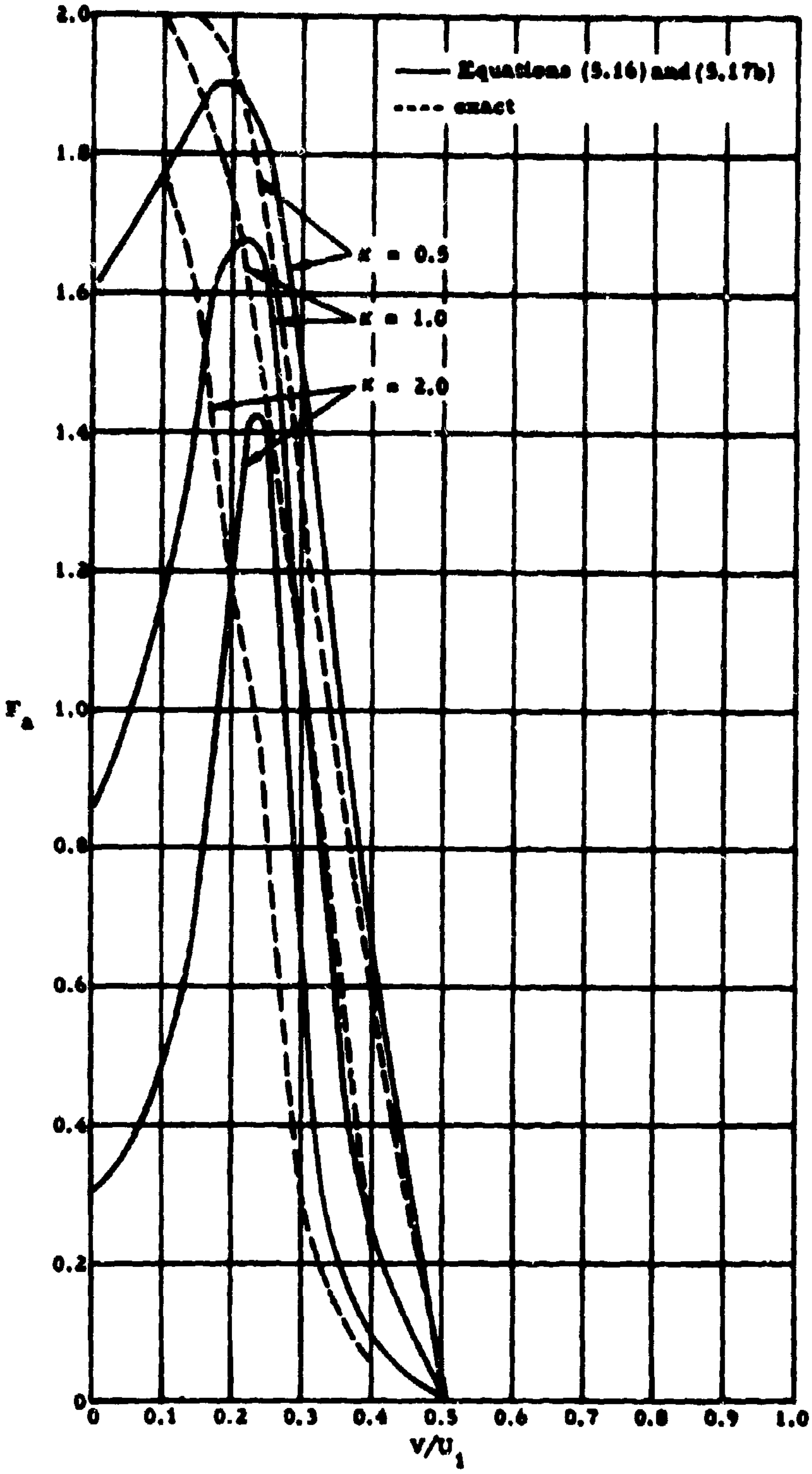

Figure 2c. F_a versus V/U_1 for M_1 = 2.0.

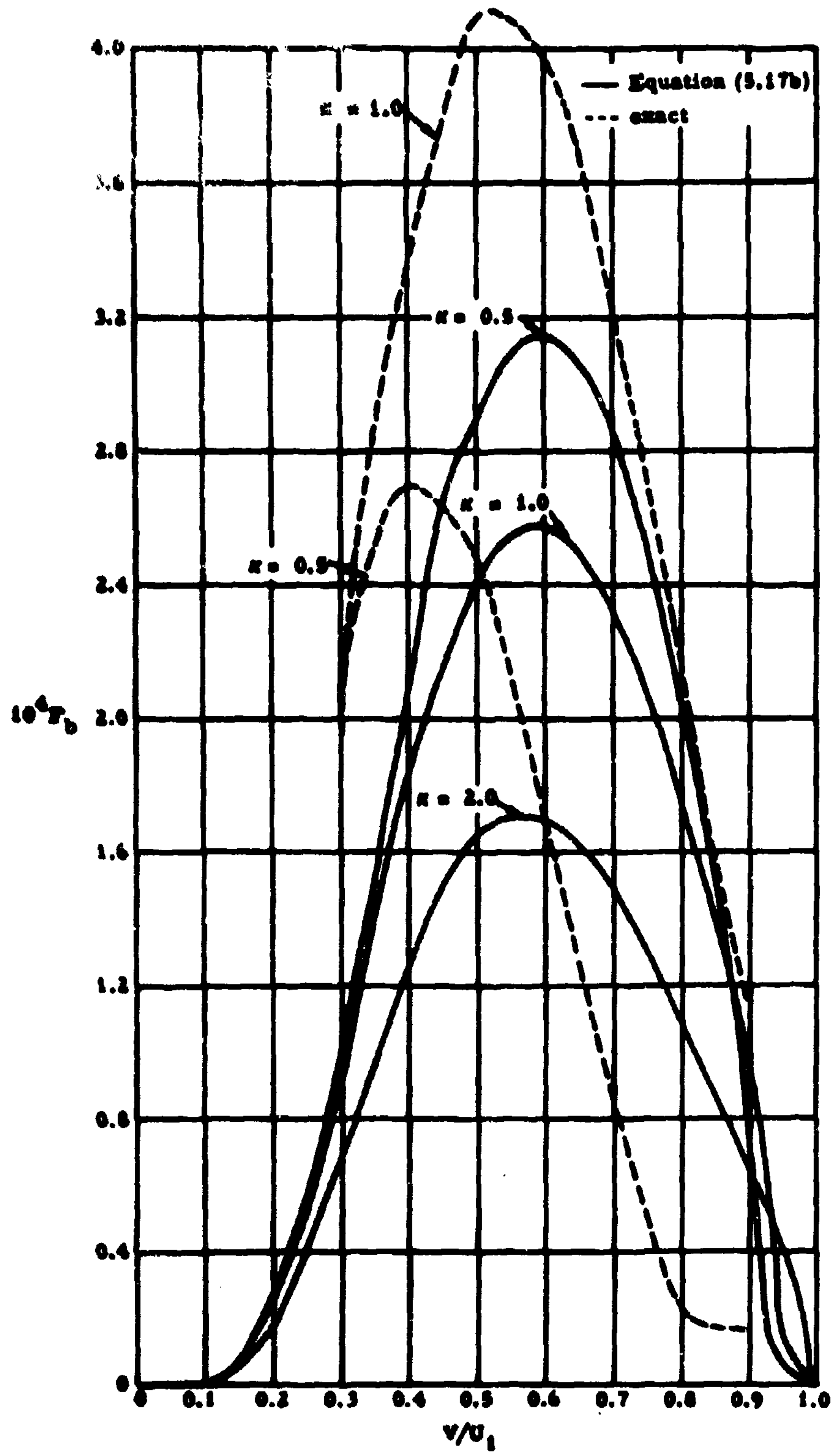

Figure 3a. F_b versus V/U_1 for $M_1 = 0.1$.

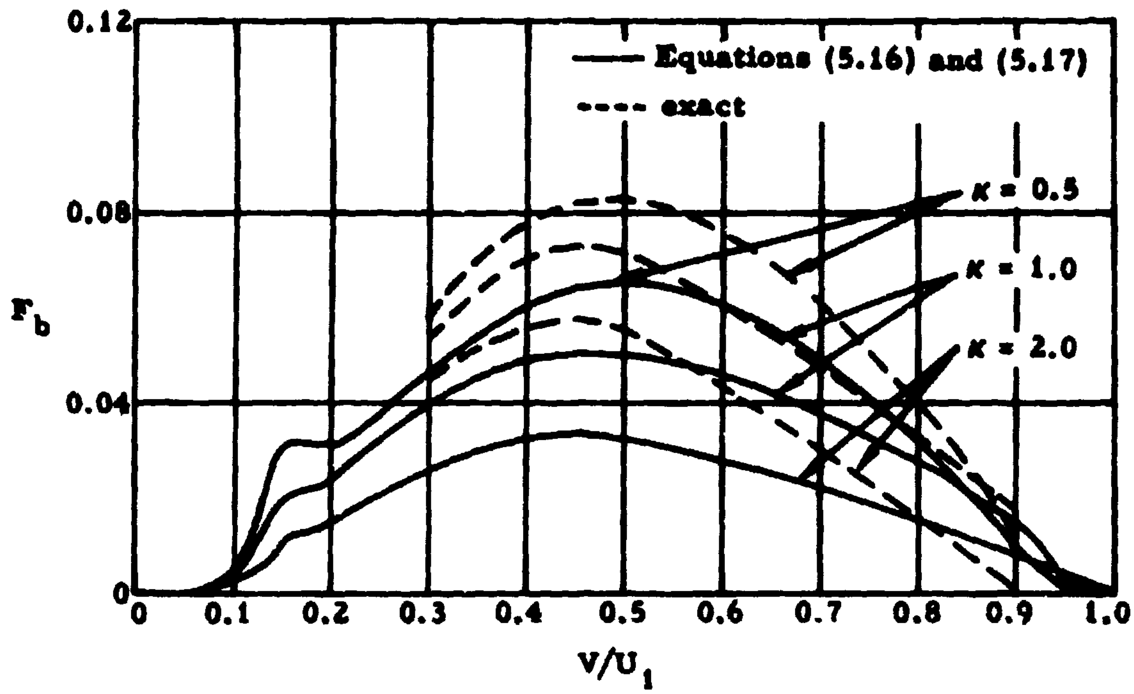

Figure 3b. F_b versus V/U_1 for $M_1 = 1.2$.

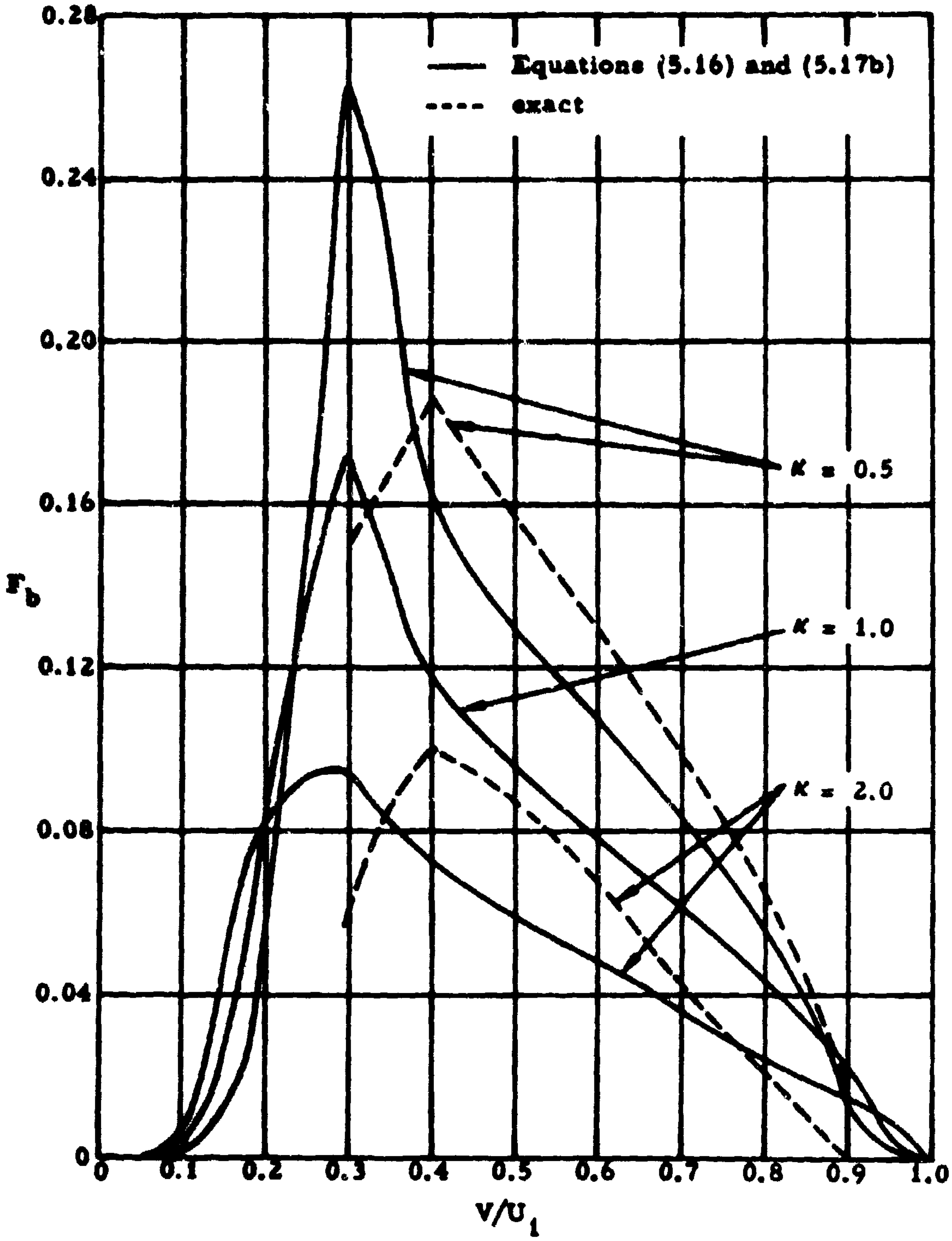

Figure 3c. F_b versus V/U_1 for $M_1 = 1.5$.

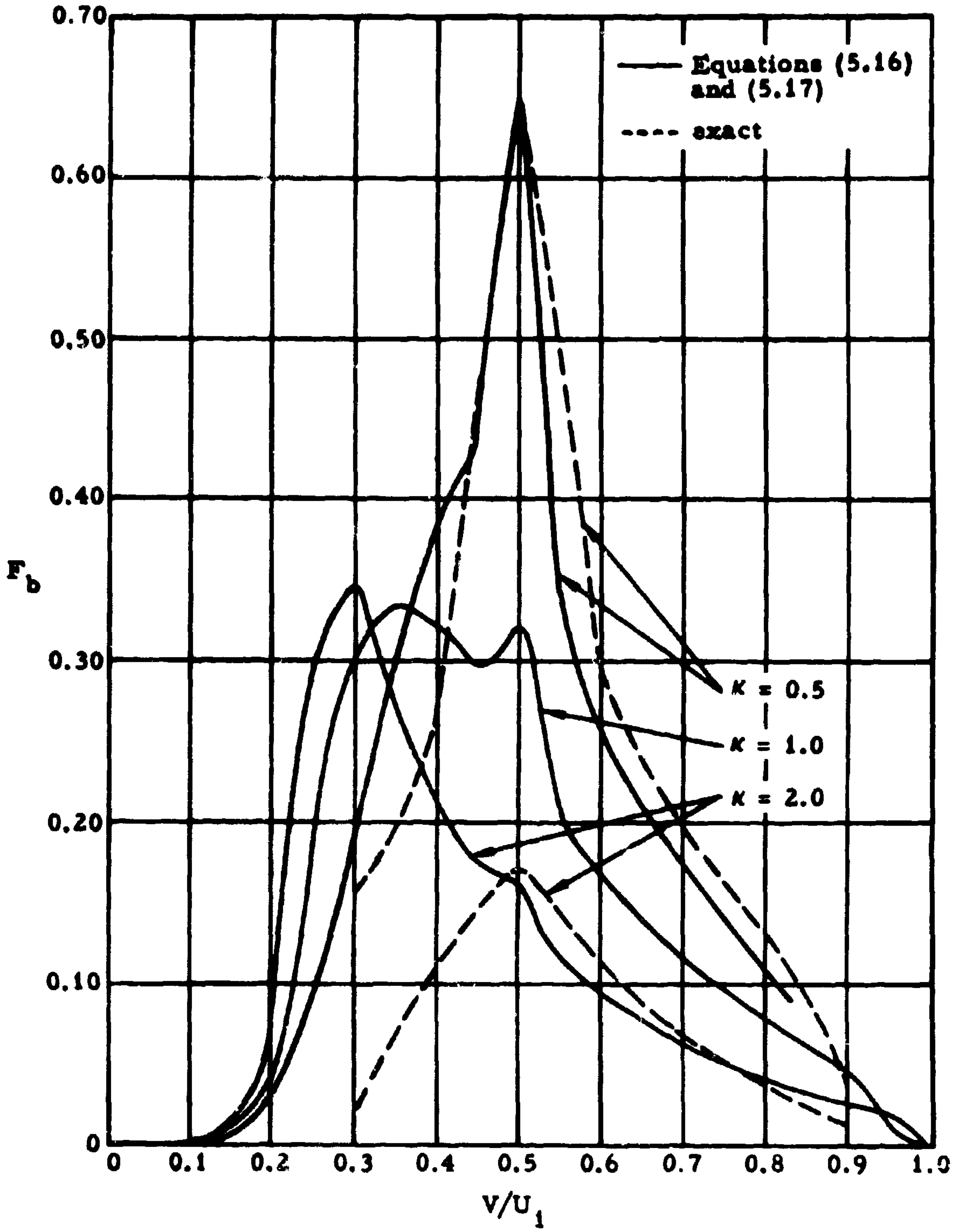

Figure 3d. F_b versus V/U_1 for $M_1 = 2.0$.

6. APPROXIMATE STABILITY ANALYSIS

We consider in this section the approximate calculation of the negative damping ratio (in the notation of reference 2)

$$\delta_o = - \operatorname{Im} V / \operatorname{R\ell} V \tag{6.1}$$

on the assumption that the second term in the braces of Eq. (4.1) is sufficiently small compared with unity to permit the approximation

$$V = V_o \left\{ 1 - \frac{1}{2} \mu \left(\frac{A}{V_o} \right)^2 F \left(\frac{U_1}{A}, \frac{V_o}{U_1} \right) \right\} . \tag{6.2}$$

Eq. (6.2) may be regarded as the first approximation based on a power series expansion in the parameter μ, and it will be valid for $\mu << 1$ insofar as F remains bounded. Now μ [see Eq. (3.5)] almost certainly would be small in practical applications, while $|F|$ can be large only if both $|V - (U_1 \pm A)| << U_1$ and $K << 1$; the latter contingency is examined in the following section, but we remark here that it is apt to be important only when K is of the same order of magnitude as $\mu^{1/3}$.

Substituting Eq. (6.2) in Eq. (6.1) and evaluating $\operatorname{Im} F$ from Eq. (4.4) yields

$$\delta_o = \frac{1}{2} \mu \left(\frac{A}{V_o} \right)^2 (F_a + F_b) , \quad 0 < V_o < U_1 - A \tag{6.3}$$

in the region of principal interest. The maximum value of δ_o with respect to V_o almost certainly will occur in this region if $(V_o)_{min} < U_1 - A$, but $F_a = 0$ if $U_1 - A < V_o < U_1$; if $V_o > U_1$ panel flutter of the type considered here is not possible.

Perhaps the most expedient approach to the determination of the maximum value of δ_o with respect to variations of K and α is to adopt the approximations of Eqs. (5.16) and (5.17b) for F_a and F_b, determine the corresponding value of $(\delta_o)_{max}$, and then estimate the approximate corrections to F_a and F_b from the results of Fig. 2. Substituting Eqs. (5.16) and (5.17b) in Eq. (6.3) yields

$$\delta_o = \frac{1}{2}\mu\left(\frac{A}{V_o}\right)^2 m_1^2 \,\mathrm{Im}\left\{\frac{1}{\sqrt{1-m_1^2} + K m_1^2 K_1}\right\} \tag{6.4a}$$

$$= \frac{1}{2}\mu\left(\frac{U_1}{V_o} - 1\right)^2 \frac{\left[\sqrt{m_1^2 - 1} - K m_1^2 \,\mathrm{Im}\, K_1\right]}{\left[\sqrt{m_1^2 - 1} - K m_1^2 \,\mathrm{Im}\, K_1\right]^2 + \left[K m_1^2 \,\mathrm{R}\ell\, K_1\right]^2}, \quad m_1 > 1. \tag{6.4b}$$

We recall that [Eq. (2.9)]

$$m_1 = \left(\frac{U_1 - V_o}{a_1}\right) \cos\alpha = M_1 \left(1 - \frac{V_o}{U_1}\right) \cos\alpha , \tag{6.5}$$

while K_1 depends on V_o/U_1, M_1, and $\cos\alpha$; as suggested in Appendix B, the error in assuming that K_1 depends only on $M_1 \cos\alpha$, rather than M_1 and $\cos\alpha$ separately, usually will be small (the essential approximation is the replacement of M_1 by $M_1 \cos\alpha$ in the temperature profile).

We consider as a more specific example a pressurized cylindrical shell for which the internal pressure is sufficiently high to render negligible the effects of bending on V_o, which then is given by [Ref. 2, Eq. (5.5) with $D = 0$ and $N_y = 2N_x = p_i R$]

$$V_o^2 = \frac{1}{\sigma}\left[Eh\left(\frac{\cos\alpha}{kR}\right)^2 + p_i R\left(\frac{1}{2} + \tan^2\alpha\right)\right] . \tag{6.6}$$

We introduce as the independent variables

$$x = V_o/U_1, \qquad y = \cos^2\alpha \tag{6.7a, b}$$

and, as a measure of the wavelength,

$$z = z_1/kR, \qquad z_1 = \sqrt{2Eh/p_i R} \quad . \tag{6.8a, b}$$

Substituting (6.7a, b) and (6.8a, b) in (6.6), we obtain

$$z = \sqrt{\frac{1}{y}\left[1 + \left(\frac{x}{x_m}\right)^2\right] - \frac{2}{y^2}} \, , \tag{6.9}$$

where

$$x_m = \sqrt{p_i R/2\sigma\, U_1^2} \tag{6.10}$$

denotes the minimum value of x ($x = x_m$ at $\alpha = 0$ and $k = \infty$). The boundary layer thickness and aerodynamic mass parameters [the latter given by Eq. (3.5) with $\rho_i = 0$ therein] then may be expressed as

$$K = k\delta = z_1\,(\delta/R)\, z^{-1} \tag{6.11}$$

and

$$\mu = \rho_1/\sigma k = (\rho_1 R/\sigma z_1) z \ . \tag{6.12}$$

Noting also that

$$m_1^2 = M_1^2\,(1-x)^2\, y \tag{6.13}$$

and, on the basis of the aforementioned assumption for K_1 ,

$$m_1^2\, K_1 = m_1^2\, K_1\,(V_o/U_1, M_1 \cos\alpha) = m_1^2\, K_1\,(x, M_1 y^{1/2}) \ , \tag{6.14}$$

Eq. (6.4b) may be transformed to

$$\delta_o = \frac{i}{2}\left(\frac{\rho_1 R}{\sigma z_1}\right) z \left(\frac{1-x}{x}\right)^2 \ \mathrm{Im}\left\{\frac{1}{-i\sqrt{M_1^2(1-x)^2 y^2 - 1 + z_1(\delta/R)z^{-1} m_1^2 K_1(x, M_1 y^{1/2})}}\right\} . \tag{6.15}$$

We require the maximum value of δ_o, as given by Eq. (6.15) in conjunction with Eq. (6.9), with respect to independent variations of x and y over the intervals

$$x_m \leq x \leq 1 \qquad \text{and} \qquad \frac{2}{1+(x/x_m)^2} \leq y \leq 1 \ . \tag{6.14a, b}$$

The required calculations would be difficult to carry out analytically, but they are quite simple on a high-speed computer.

If the boundary layer is relatively thin ($K \ll 1$) we find that $(\delta_o)_{max}$ is likely to occur for values of x and y such that m_1 exceeds unity by only a small amount and the two terms in the braces of Eq. (6.15) are approximately equal in magnitude. The imaginary part of K_1--and, therefore, the negative damping effect of profile curvature--then is likely to prove rather unimportant, although its real part plays an essential role in preventing the infinity that would be indicated by Eq. (6.15) at $m_1 = 1$ if $K \rightarrow 0^*$ (in which case the analysis of the following section would be required). The approximations of Eqs. (5.16) and (5.17b) evidently are entirely adequate for this case.

If the boundary layer is relatively thick (say $K > 1$ or 2) and x_m is not too small, it appears that $(\delta_o)_{max}$ is likely to occur at $m_1 = 1$ and $a = 0$. The instability in this case would be associated entirely with profile curvature, and it therefore would be important to improve the approximation to F_b, relative to that of Eq. (5.17b), and to investigate the effects of possible departures from the one-seventh power law.

Calculations based on the results of this section have been carried out for pressurized, monocoque shells filled with either gas or liquid and compared with calculations based on the results of reference 2. It appears that the degree of instability for supersonic flutter may be reduced by an order of magnitude (roughly a factor of 5 - 10) in consequence of boundary layers for which $K \geq 1/2$.

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* The effect of the boundary layer then may be compared with that of small damping in limiting the response of a simple oscillator at resonance.

7. STABILITY ANALYSIS—VERY THIN BOUNDARY LAYER

The analysis of the preceding section breaks down in the neighborhood of $m_1 = 1$ if K is of the same order as $\mu^{1/3}$ (of course, if K is sufficiently small compared with μ the effects of the boundary layer on panel flutter will be negligible, but turbulent boundary layers are not apt to be this thin at supersonic speeds). The assumption that both K and $|m_1 - 1|$ are small allows us to approximate F by [see Eqs. (5.11) and (A 8a)-(A 9b); also cf. Eqs. (5.16) and (5.17a), with which this approximation is consistent]

$$F = \frac{m_1^2}{\left[(1 - m_1^2)^{1/2} + K m_1^2 K_1\right]} \left[1 + O(K^2)\right]. \tag{7.1}$$

Substituting F from (7.1) in (3.4) and eliminating m_1 through (2.9) and (2.18), we obtain, after some algebraic manipulation,

$$\left(\frac{V^2 - V_o^2}{A^2}\right)\left\{\left[1 - \left(\frac{U_1 - V}{A}\right)^2\right]^{1/2} + K\left(\frac{U_1 - V}{A}\right)^2 K_1\left(\frac{U_1}{A}, \frac{V}{U_1}\right)\right\} + \mu\left(\frac{U_1 - V}{A}\right)^2 = 0 , \tag{7. 2}$$

where the dependence of K_1 on both U/A and V/U_1 has been explicitly denoted.

We next introduce the dimensionless variables ν and ξ, representing the departures of V and V_o from $U_1 - A$, according to (cf. reference 2)

$$V = (U_1 - A)\left[1 + (1/2)\nu(\mu/2)^{2/3}(M_a - 1)^{-5/3}\right], \quad -3\pi/2 < \arg \nu \leq \pi/2 \tag{7.3a}$$

and

$$V_o = (U_1 - A)\left[1 + (3/2)\xi\,(\mu/2)^{2/3}\,(M_a - 1)^{-5/3}\right], \tag{7.3b}$$

where the permissible range of $\arg \nu$ is deduced from the restrictions on V and β [Eq. (2.13)], ξ is real (structural damping being neglected), and

M_α denotes the (free stream) Mach number relative to an observer moving with the wave front--viz.,

$$M_\alpha = U_1/A = U_1 \cos \alpha / a_\infty . \tag{7.4}$$

We also find it convenient to introduce the boundary layer parameters θ and η according to

$$\theta = \left(\frac{K}{3}\right)\left[2\left(\frac{M_\alpha - 1}{\mu}\right)\right]^{1/3} K_1 (M_\alpha , \; 1 - M_\alpha^{-1}) \tag{7.5a}$$

and

$$\eta = 3^{1/2}\theta . \tag{7.5b}$$

We will use the parameter θ initially, but we find η more convenient in the end results. Substituting Eqs. (7.3a,b) and (7.5a) in Eq. (7.2) and neglecting higher powers of μ then yields

$$(\nu - 3\xi)(\nu^{1/2} + 3\theta) + 2 = 0. \tag{7.6}$$

We observe that the importance of the boundary layer in this approximation is determined by the size of $3|\theta|$ relative to $\nu^{1/2}$; in particular, $K << 1$ is not a sufficient condition for the neglect of the boundary layer.

Rewriting Eq. (7.6) in the form

$$(\nu^{1/2} + \theta)^3 - 3\,AB(\nu^{1/2} + \theta) + (A^3 + B^3) = 0 , \tag{7.7}$$

where

$$\begin{matrix} A \\ B \end{matrix} = \left\{(1 - 3\xi\theta + \theta^3) \pm \left[(1 - 3\xi\theta + \theta^3)^2 - (\xi + \theta^2)^3\right]^{1/2}\right\}^{1/3} , \tag{7.8}$$

we have

$$\nu^{1/2} + \theta = A\epsilon + B\bar{\epsilon} , \qquad \epsilon^3 = -1 , \tag{7.9}$$

where the three roots of the cubic equation (7.7) correspond to the three cube roots (ϵ) of -1 ($\bar{\epsilon}$ is the complex conjugate of ϵ). This result is valid for complex values of both ξ and θ, but the explicit determination of the imaginary part of ν, on which the damping ratio depends (assuming μ to be approximately real), is possible only if both ξ and θ are assumed to be real; on this last assumption we find

$$\operatorname{Im}\nu = -(\sqrt{3}/2)\left|A - B\right| \cdot \left|A + B - 2\theta\right| \qquad (1 - 3\xi\theta + \theta^3) \gtrless (\xi + \theta^3)^2 \tag{7.10a}$$

$$= 0. \tag{7.10b}$$

If θ is not real $\operatorname{Im}\nu$ must be determined numerically from Eq. (7.9).

The maximum value of $-\operatorname{Im}\nu$ with respect to the spectral parameter ξ is determined in Appendix C. We designate the ratio of this maximum to its value in the absence at the boundary layer ($3^{1/2}/2^{1/3}$ at $\theta = 0$ and $\xi = 0$) as $h(\eta)$, which is given by Eqs. (C7) and (C8) and is plotted in Fig. 4; ξ_m, the value of ξ at which this maximum occurs, is plotted in Fig. 5. The maximum (with respect to ξ) negative damping ratio then is given by [cf. Eq. (4.13), reference 2]

$$\delta_o = (-\operatorname{Im} V/R\ell V)_{\xi = \xi_m} = (3^{1/2}/4)\mu^{2/3}(M_a - 1)^{-5/3}\, h(\eta). \tag{7.11}$$

We emphasize that the results for $\eta < 0$ may not be directly significant, since K_1 and η will be positive real if the singularity at $z = z_c$ is excluded, while if it is included K_1 and η must be complex; however, these results may be useful in determining a first approximation to δ_o for complex η.

We find, by numerical comparison, that the approximations $\xi_m = -\eta^2$ and $h = (2^{1/3}/3\eta)$ are adequate for $\eta > 1$ [Eqs. (C10) and (C11)]. Substituting these approximations in Eqs. (7.3b) and (7.11) and eliminating η via Eq. (7.5) yields

$$\delta_o = \mu/4KK_1(M_a - 1)^2, \qquad \eta > 1 \tag{7.12}$$

$$V_o \Big|_{\xi = \xi_m} = (U_1 - A) - \frac{1}{2}(KK_1)^2 A \;, \qquad \eta > 1 \;. \tag{7.13}$$

The real and imaginary parts of K_1, as evaluated in Appendix B, are plotted in Fig. 6 for $V/U_1 = 1 - M_1^{-1}$ (cf. Fig. 1). As suggested above, the error in replacing M_1 by $M_1 \cos \alpha$ for $\alpha \neq 0$ is not likely to be large.

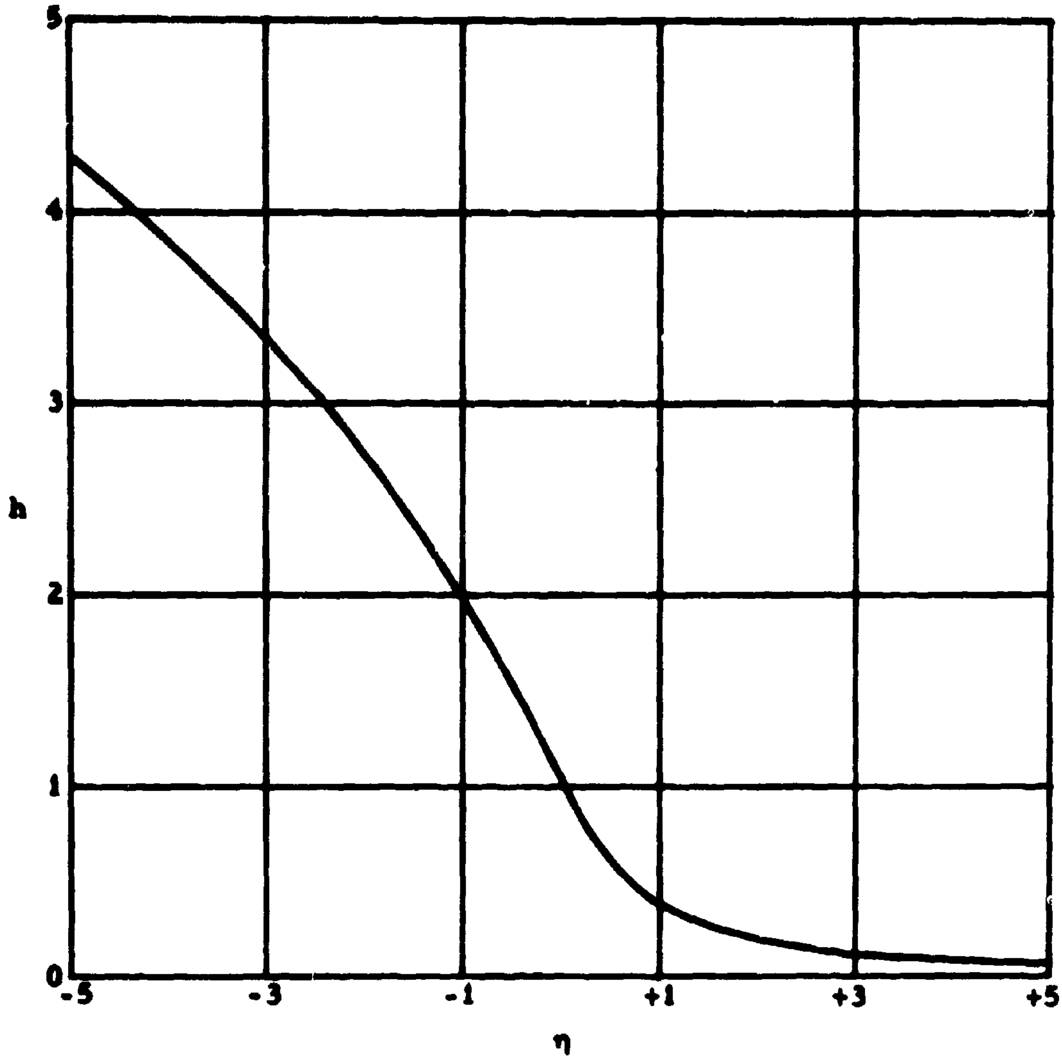

Figure 4a. $h(\eta)$, the ratio of the maximum negative damping ratio with boundary layer to that without, as given by Equations (7·11), (C7), and (C8).

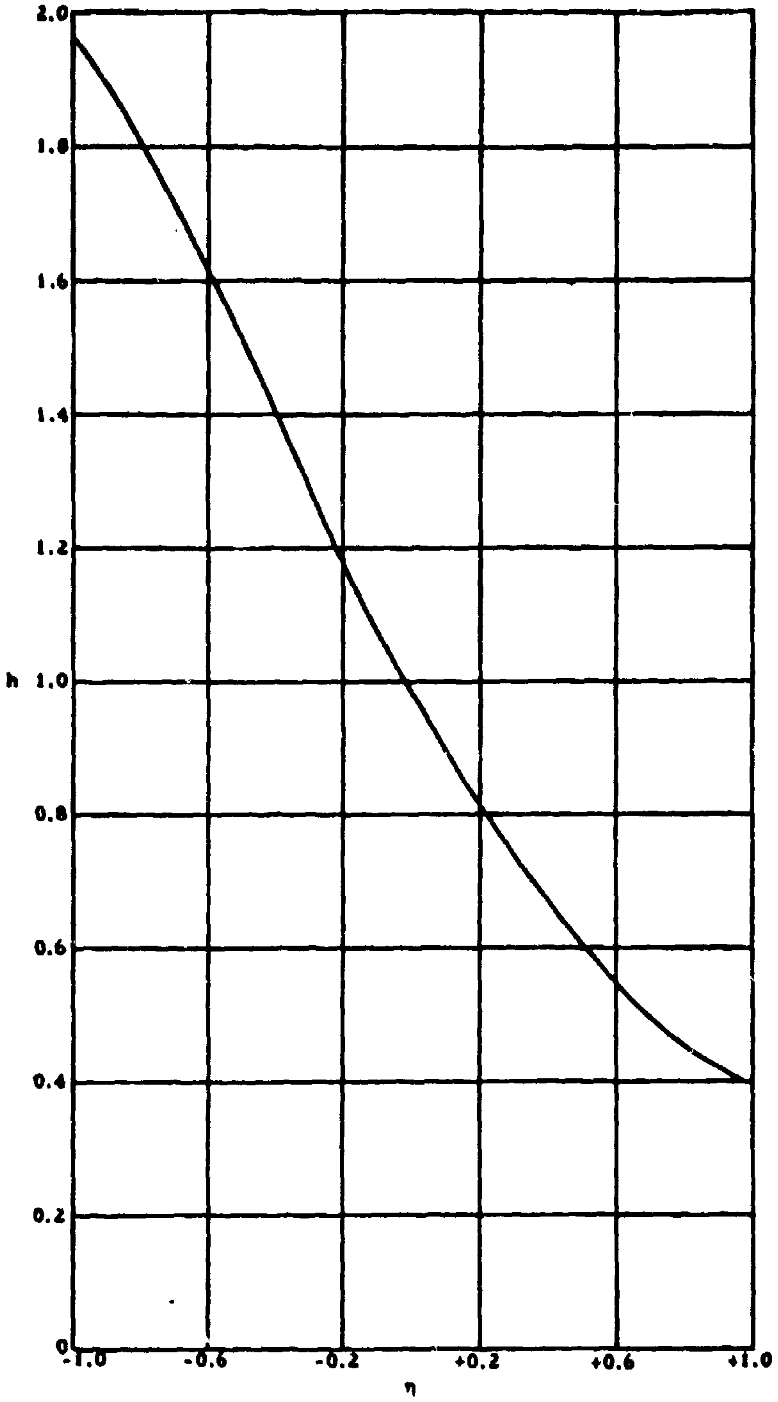

Figure 4b. Expanded plot of $h(\eta)$ for $|\eta| < 1$

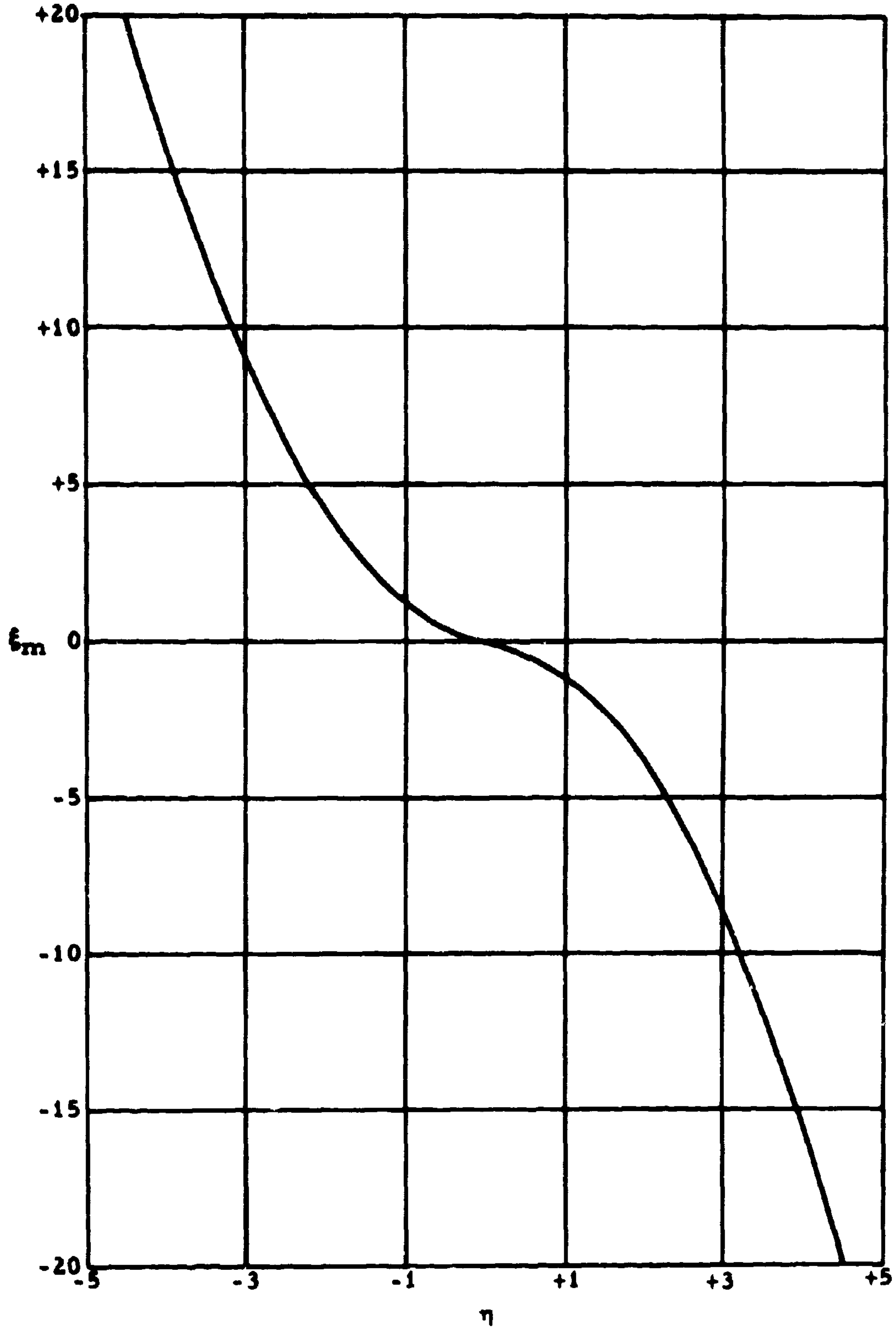

Figure 5a. ξ_m, the value of ξ for maximum negative damping ratio, as given by Equation (C8).

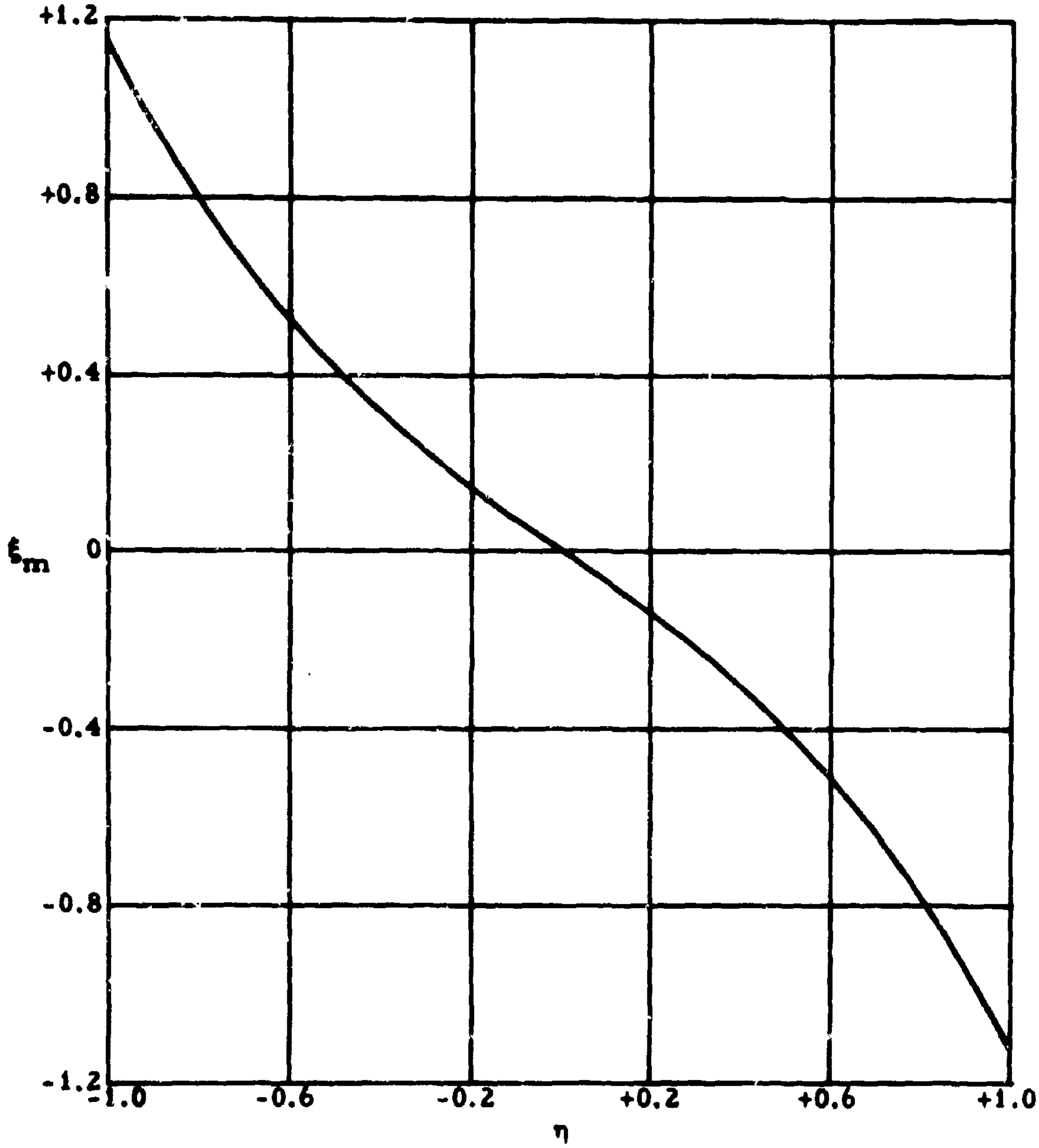

Figure 5b. Expanded plot of ξ_m for $|\eta| < 1$.

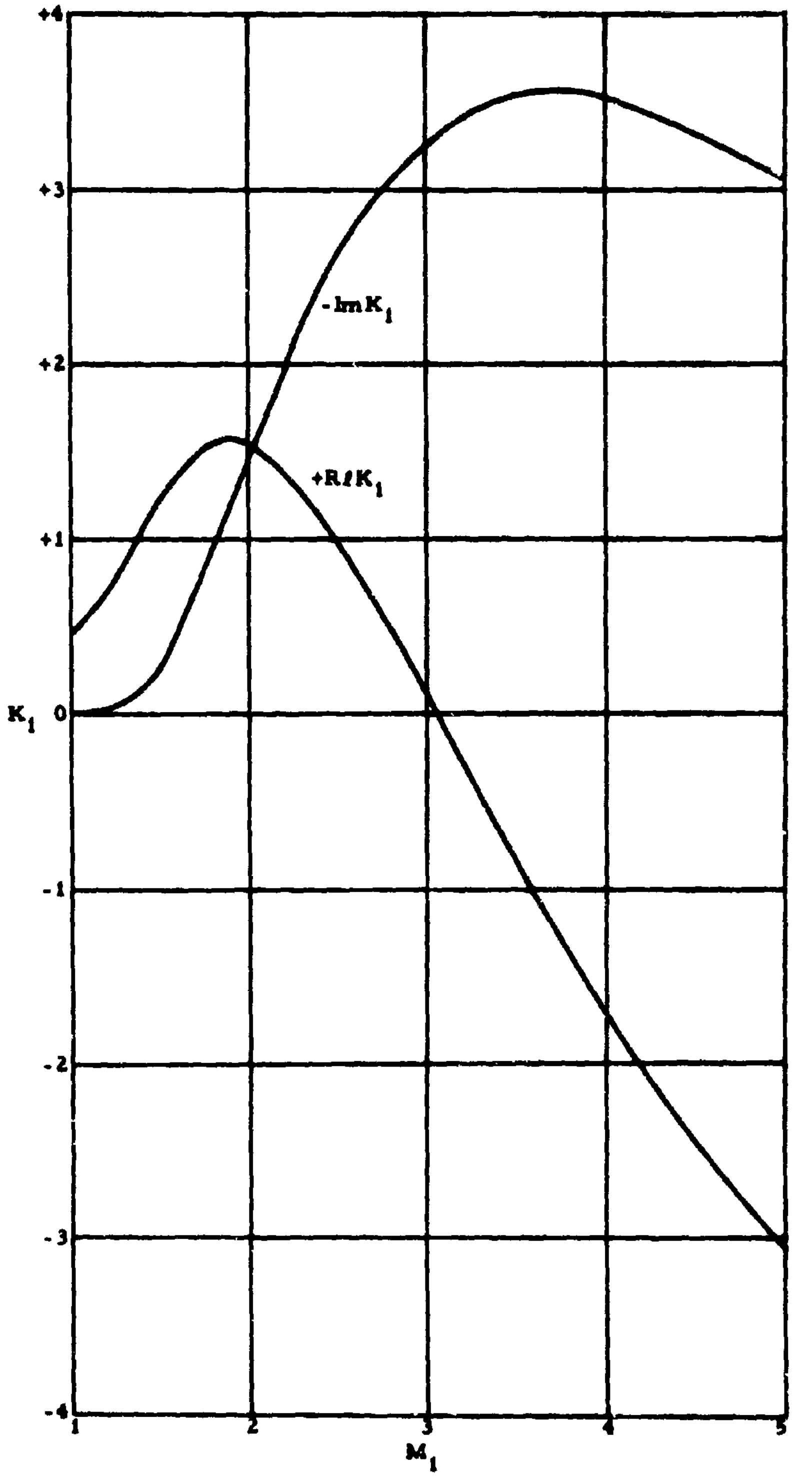

Figure 6. The real and imaginary parts of K_1 for the special case $\alpha = 0$ ($V = U_1 - a_1$) and an insulated wall with $\gamma = 1.4$.

APPENDIX A

SOLUTION OF DIFFERENTIAL EQUATION

We require solutions to Eqs. (5.7) and (5.8)--viz.,

$$m^2 \frac{d}{d\zeta}\left(m^{-2}\frac{df}{d\zeta}\right) - \kappa^2(1-m^2)\,f = 0, \qquad 0 \leq \zeta \leq 1, \tag{A1}$$

$$f_{10} = 1, \quad f'_{10} = 0, \tag{A2a}$$

$$f_{20} = 0, \quad f'_{20} = m_0^2, \tag{A2b}$$

where the second subscript indicates the value of ζ.

A solution that is useful for small or moderate values of κ may be obtained by introducing the expansions

$$f_i(\zeta;\kappa^2) = \sum_{n=0}^{\infty} \kappa^{2n} f_i^{(n)}(\zeta), \qquad i = 1, 2 \tag{A3}$$

in Eq. (A1), which then yields

$$\frac{d}{d\zeta}\left(m^{-2}\frac{df_i^{(n)}}{d\zeta}\right) = (m^{-2}-1)\,f_i^{(n-1)}. \tag{A4}$$

We may satisfy the boundary conditions of Eq. (A2) with the zero'th approximations

$$f_1^{(0)} = 1 \tag{A5a}$$

and

$$f_2^{(0)} = \int_0^{\zeta} m^2\,d\zeta. \tag{A5b}$$

The succeeding terms ($n \geq 1$) in Eq. (A3) then must satisfy the homogeneous boundary conditions

$$f_i^{(n)}(0) = f_i^{(n)'}(0) = 0. \qquad \text{(A6)}$$

Integrating Eq. (A4) subject to these conditions, we obtain the recursion formula

$$f_i^{(n)} = \int_0^{\zeta} m^2 d\zeta \int_0^{\zeta} (m^{-2} - 1)\, f_i^{(n-1)} d\zeta, \qquad n \geq 1. \qquad \text{(A7)}$$

We require, in the end [cf. Eq. (2.24)], the values of f_1, f_1', f_2, and f_2' only at $\zeta = 1$. Following Lees and Lin[6,8] (but with slightly revised notations), we may express these in a series of definite integrals according to

$$f_{11} = \sum_{n=0}^{\infty} K^{2n} K_{2n}, \qquad \text{(A8a)}$$

$$f_{11}' = m_1^2 \sum_{n=0}^{\infty} K^{2n+2} K_{2n+1}, \qquad \text{(A8b)}$$

$$f_{21}' = m_1^2 \sum_{n=0}^{\infty} K^{2n} H_{2n}, \qquad \text{(A9a)}$$

$$f_{21} = \sum_{n=0}^{\infty} K^{2n} H_{2n+1}, \qquad \text{(A9b)}$$

$$K_o = 1,$$

$$K_1 = \int_0^1 (m^{-2} - 1)\, d\zeta,$$

$$K_2 = \int_0^1 m^2 d\zeta \int_0^{\zeta} (m^{-2} - 1)\, d\zeta,$$

$$K_3 = \int_0^1 (m^{-2} - 1)\, d\zeta \int_0^{\zeta} m^2 d\zeta \int_0^{\zeta} (m^{-2} - 1)\, d\zeta, \; \ldots\; , \qquad \text{(A10)}$$

$$H_0 = 1,$$

$$H_1 = \int_0^1 m^2 d\zeta ,$$

$$H_2 = \int_0^1 (m^{-2} - 1)\, d\zeta \int_0^\zeta m^2 d\zeta, \; . \; . \; . \; . \qquad \text{(A11)}$$

The numerical evaluation of these integrals for a laminar boundary layer has been discussed in some detail by Lees.[8] Appendix B contains a discussion of the evaluation of K_1 (the most important of these integrals) for a turbulent boundary layer.

The analytic solution w_1, as introduced in Section 5, also may be expanded according to Eq. (A3). Integrating Eq. (A4) subject to the boundary conditions

$$w_1 \doteq (\zeta - \zeta_c)^3, \quad \zeta \rightarrow \zeta_c , \qquad \text{(A12)}$$

we obtain

$$w_1^{(0)} = \frac{3}{m_c'^{\,2}} \int_{\zeta_c}^{\zeta} m^2 d\zeta , \qquad \text{(A13a)}$$

$$w_1^{(n)} = \int_{\zeta_c}^{\zeta} m^2 d\zeta \int_{\zeta_c}^{\zeta} (m^{-2} - 1)\, w_1^{(n-1)} d\zeta. \qquad \text{(A13b)}$$

We require $w_1(1)$ and $w_1'(1)$, which may be placed in the forms

$$w_{11}' = \frac{3}{m_c'^{\,2}} \quad m_1^2 \quad \sum_0^\infty K^{2n} J_{2n} , \qquad \text{(A14a)}$$

$$w_{11} = \frac{3}{m_c'^{\,2}} \quad \sum_0^\infty K^{2n} J_{2n+1} . \qquad \text{(A14b)}$$

$$J_o = 1,$$

$$J_1 = \int_{\zeta_c}^{1} m^2 d\zeta,$$

$$J_2 = \int_{\zeta_c}^{1} (m^{-2} - 1)\, d\zeta \int_{\zeta_c}^{\zeta} m^2 d\zeta,$$

$$J_3 = \int_{\zeta_c}^{1} m^2 d\zeta \int_{\zeta_c}^{\zeta} (m^{-2} - 1)\, d\zeta \int_{\zeta_c}^{\zeta} m^2 d\zeta\ , \ . \ . \ . \ . \qquad \text{(A15)}$$

APPENDIX B

EVALUATION OF K_1

We consider the integral

$$K_1 = \delta^{-1}\int_0^{\delta} (m^{-2} - 1)\, dz, \quad m(z) = \frac{U(z) - V}{a(z)\sec\alpha} . \tag{B1}$$

Introducing the dimensionless notation

$$\zeta = \frac{z}{\delta},\ \xi = \frac{U}{U_1},\ \xi_c = \frac{V}{U_1},\ \tau = \tau(\xi) = \frac{a^2(z)}{a_1^2}, \tag{B2}$$

where $\tau(\xi)$ is the relative temperature distribution, expressed as a function of the relative velocity, K_1 becomes

$$K_1 = -1 + M_1^{-2}\sec^2\alpha\, I(\xi_c), \tag{B3}$$

where

$$I(\xi_c) = \int_0^1 \frac{\tau\, d\zeta}{(\xi - \xi_c)^2} \tag{B4a}$$

$$= \int_0^1 \frac{\tau(\xi)\,\zeta'(\xi)\, d\xi}{(\xi - \xi_c)^2}, \tag{B4b}$$

and we have assumed $U = U_1$ ($\xi = 1$) at $z = \delta$ ($\zeta = 1$).

Integrating Eq. (B4b) once by parts, along a path indented over the singularity at $\zeta = \zeta_c$, and then separating out the singular portion (I_s), we obtain

$$I = -\tau(0)\zeta'(0)\xi_c^{-1} - \zeta'(1)(1 - \xi_c)^{-1} + I_s + \int_0^1 \frac{\left[(\tau\zeta')' - (\tau\zeta')'_c\right] d\xi}{(\xi - \xi_c)} \tag{B5}$$

where

$$I_s = (\tau\zeta')'_c \int_0^1 \frac{d\xi}{(\xi - \xi_c)} , \tag{B6}$$

and the subscript c implies evaluation at the singular point $\xi = \xi_c$. Integrating I_s over the singular point yields directly

$$I_s = (\tau\zeta')'_c \left[\ln \left(\frac{1 - \xi_c}{\xi_c} \right) - \pi i \right] . \tag{B7}$$

Substituting this result in Eq. (B1) yields for the imaginary part of K_1

$$\operatorname{Im} K_1 = - \pi M_1^{-2} \sec^2 \alpha \, (\tau\zeta')'_c \tag{B8a}$$

$$= - \pi M_1^{-2} \sec^2 \alpha \, \delta^{-1} \left[\frac{U_1^2}{U'(z_c)} \right] \frac{d}{dz} \left[\frac{\tau}{U'(z)} \right]_{z=z_c} \tag{B8b}$$

$$= \pi \delta^{-1} a_c^2 \sec^2 \alpha \left[\frac{T(z_c)}{U'^3(z_c)} \right] \frac{d}{dz} \left[\frac{U'(z)}{T(z)} \right]_{z=z_c} . \tag{B8c}$$

We consider, as a particular example, the turbulent velocity profile

$$\zeta = \xi^7 \tag{B9}$$

together with the quadratic (in velocity) temperature profile[9]

$$\tau(\xi) = a + b\xi + c\,\xi^2, \quad a = \frac{T_o}{T_1}, \quad c = -\left(\frac{\gamma - 1}{2}\right) M_1^2 , \quad b = 1 - a - c, \tag{B10}$$

for which

$$(\tau\zeta')' = 7(6a\,\xi^5 + 7b\,\xi^6 + 8c\,\xi^7). \tag{B11}$$

We note that Eq. (B9) is not accurate right down to $\zeta = 0$, where it gives $\zeta'(0) = 0$, but the error in equating the first term in Eq. (B5) to zero for large free stream Reynolds numbers (say greater than 10^6) is negligible. Substituting Eqs. (B9) through (B11) in Eq. (B5) and carrying out the integrations yields

$$I = 7\left[a I_5(\xi_c) + b I_6(\xi_c) + c I_7(\xi_c) - (1 - \xi_c)^{-1}\right], \tag{B12}$$

where

$$I_n(\xi_c) = (n + 1)\left\{\sum_{m=0}^{n-1} \frac{\xi_c^m}{n - m} + \xi_c^n \left[I_n\left(\frac{1 - \xi_c}{\xi_c}\right) - \pi i\right]\right\}. \tag{B13}$$

Substituting Eq. (B12) in Eq. (B3) and a, b, and c from Eq. (B10) then yields

$$\begin{aligned} K_1 = -1 + 7M_1^{-2} \sec^2 \alpha &\left\{\left[I_6(\xi_c) - (1 - \xi_c)^{-1}\right]\right. \\ &+ \left(\frac{T_o}{T_1}\right)\left[I_5(\xi_c) - I_6(\xi_c)\right] \\ &\left. + \left(\frac{\gamma - 1}{2}\right) M_1^2 \left[I_6(\xi_c) - I_7(\xi_c)\right]\right\} . \end{aligned} \tag{B14}$$

The quantity actually required in the stability calculations is

$$m_1^2 K_1 = \left(\frac{U_1 - V_o}{A}\right)^2 K_1 = M_1^2 \cos^2 \alpha \cdot (1 - \xi_c)^2 K_1 , \tag{B15}$$

which we write in the form

$$m_1^2 K_1 = -m_1^2 + \left(\frac{T_o}{T_1}\right) A(\xi_c) + B(\xi_c) + \left(\frac{\gamma - 1}{2}\right) M_1^2 \; C(\xi_c) \tag{B16}$$

with

$$A(\xi_c) = 7(1 - \xi_c)^2 \left[I_5(\xi_c) - I_6(\xi_c)\right] \tag{B17}$$

$$B(\xi_c) = 7(1-\xi_c)\left[(1-\xi_c)\, I_6(\xi_c) - 1\right] \tag{B18}$$

$$C(\xi_c) = 7(1-\xi_c)^2\left[I_6(\xi_c) - I_7(\xi_c)\right] . \tag{B19}$$

The real and imaginary parts of A, B, and C are plotted in Figs. 7a, b, c, respectively. The real and imaginary parts of $m_1^2 K_1$ are plotted in Figs. 1a, b for $\alpha = 0$, $M_1 = 0.1$, 1.2, 1.5, and 2.0, and an insulated boundary--i.e., one for which

$$\frac{T_o}{T_1} = 1 + \left(\frac{\gamma - 1}{2}\right) M_1^2 \quad . \tag{B20}$$

The latter results also may be used, with only a small error, for $\alpha \neq 0$ if the nominal value of M_1 on the curves is replaced by $M_1 \cos \alpha$.

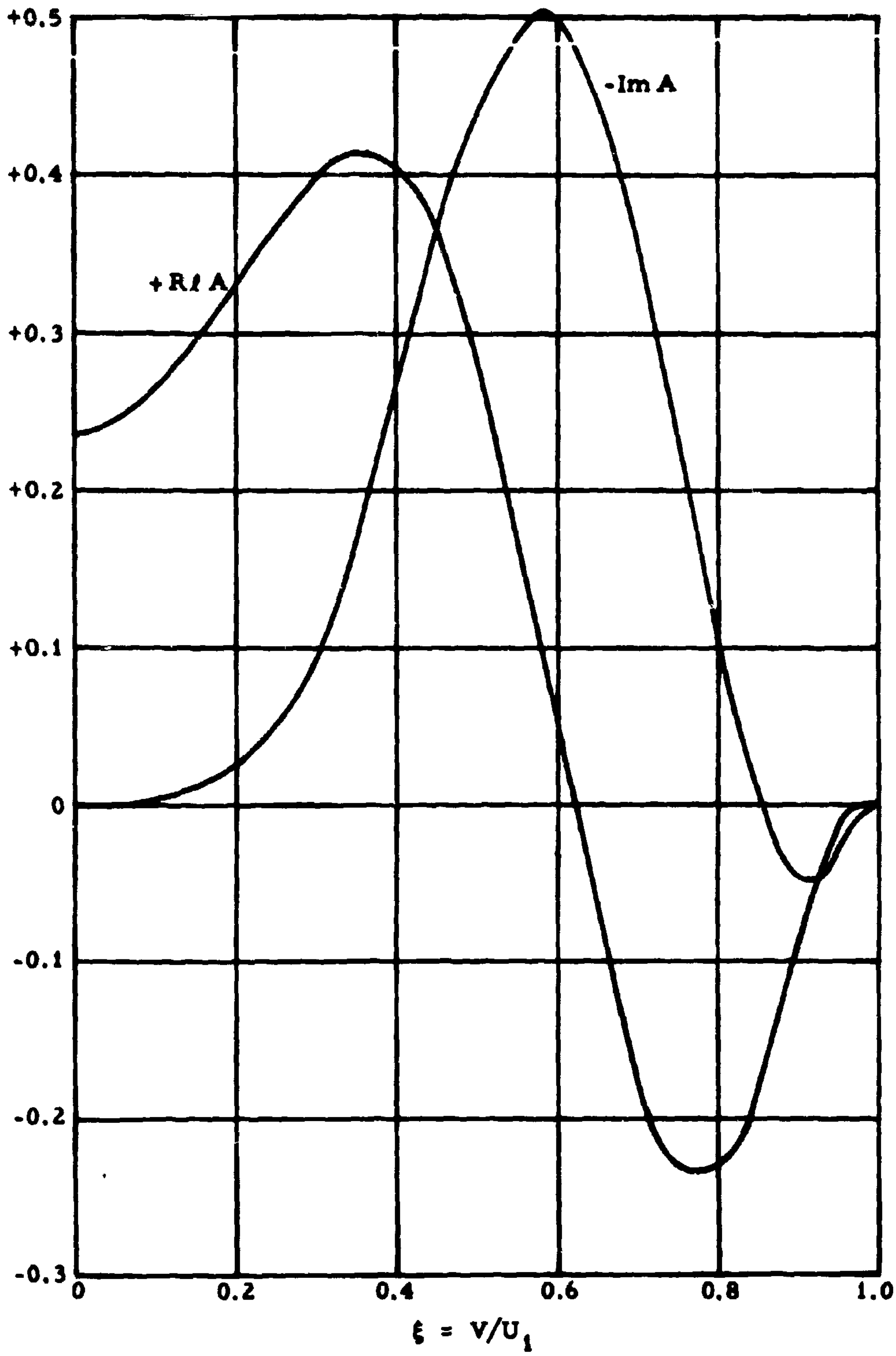

Figure 7a. A (ξ), as given by Equation (B17).

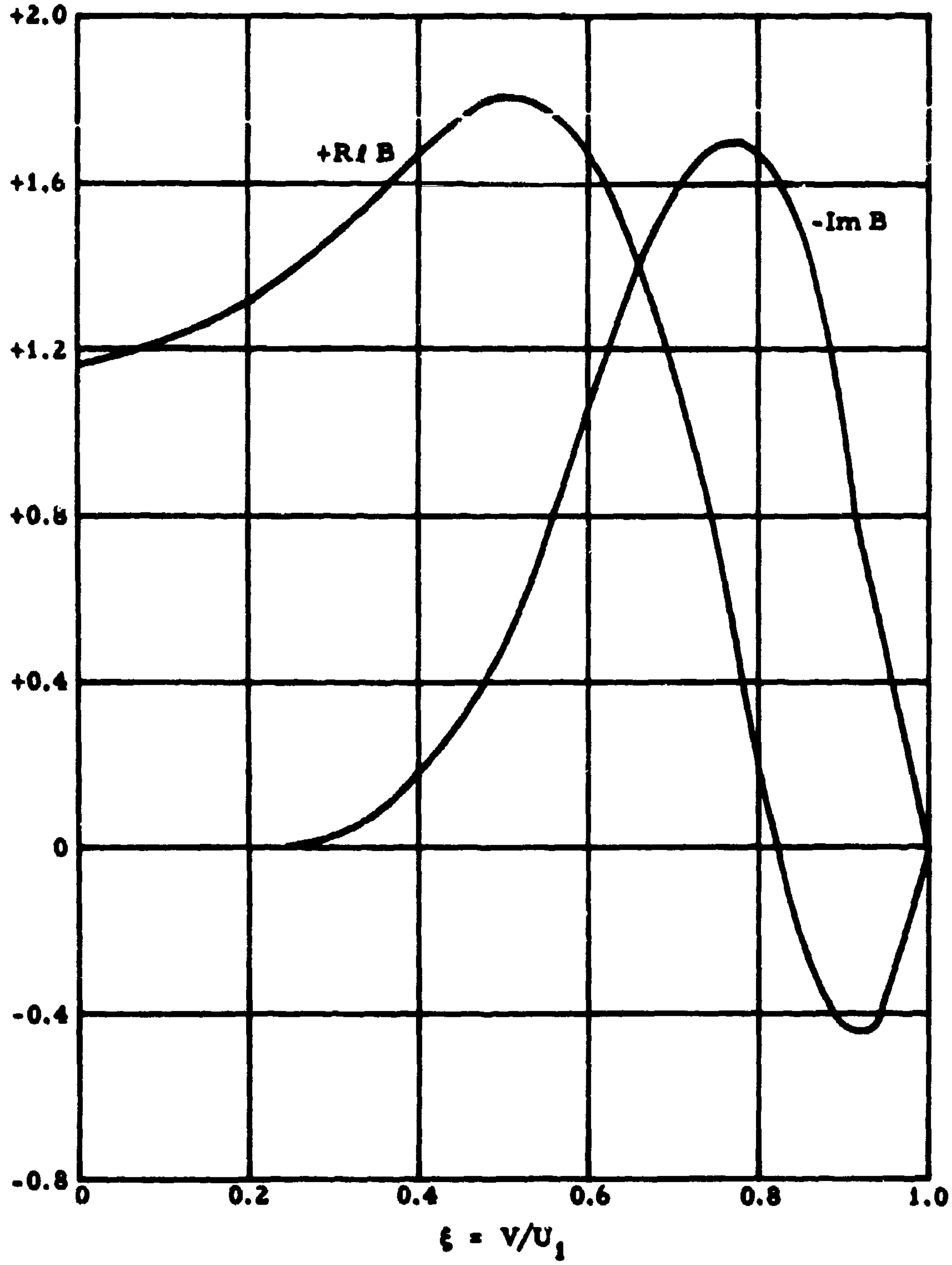

Figure 7b. B (ξ), as given by Equation (B18).

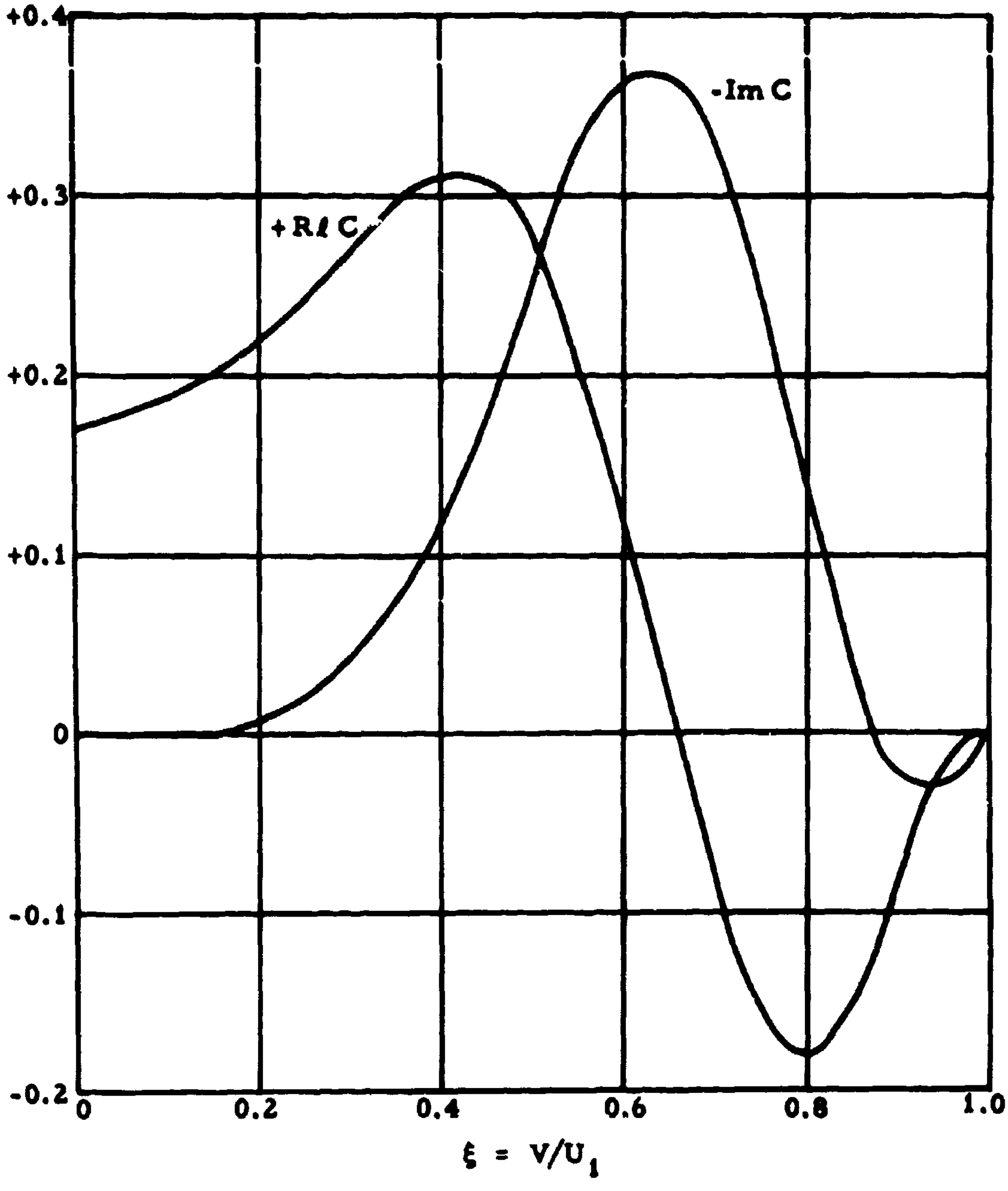

Figure 7c. C (ξ), as given by Equation (B19).

APPENDIX C

MAXIMUM VALUE OF Im ν

We wish to maximize the quantity

$$- \text{Im}\, \nu = (\sqrt{3}/2) \left| A - B \right| \cdot \left| A + B - 2\theta \right| \quad \text{(C1)}$$

with respect to ξ, A and B being given by

$$\begin{matrix} A \\ B \end{matrix} = \left\{ (1 - 3\xi\theta + \theta^3) \pm \left[(1 - 3\xi\theta + \theta^3)^2 - (\xi + \theta^2)^3 \right]^{1/2} \right\}^{1/3} . \quad \text{(C2)}$$

Differentiating Eq. (C1) with respect to ξ and equating the result to zero, we obtain

$$\left[(A + B)\theta + AB \right] (A + B - 2\theta) = 0 . \quad \text{(C3)}$$

It is evident that the maximum value of $-\text{Im}\, \nu$ corresponds to the vanishing of the first factor in Eq. (C3); dividing this through by $AB\theta$ and combining the resulting equation with the values of AB and $A^3 + B^3$, obtained from Eq. (C1), we have

$$A^{-1} + B^{-1} + \theta^{-1} = 0 , \quad \text{(C4a)}$$

$$AB - (\xi + \theta^2) = 0 , \quad \text{(C4b)}$$

and

$$A^3 + B^3 - 2(1 - 3\xi\theta + \theta^3) = 0 . \quad \text{(C4c)}$$

Eliminating A and B from Eqs. (C4a, b, c) yields the cubic equation

$$\xi^3 - 9\theta^4\xi + 2\theta^3 = 0 \quad \text{(C5)}$$

for the required value of ξ, say ξ_m. The corresponding value of $-\text{Im}\, \nu$ may be placed in the form

$$(-\text{Im}\, \nu)_{max} = 3^{1/2} \cdot 2^{-1/3} h(\eta), \qquad \eta = 3^{1/2}\theta , \quad \text{(C6)}$$

where h, normalized to have the value unity at $\eta = 0$, is given by

$$h = 3 \cdot 2^{-2/3} \eta^{-2} \cdot \left| \xi + \eta^2 \right| \cdot \left[(\xi - \eta^2)(\xi + \tfrac{1}{3}\eta^2) \right]^{1/2} , \quad \text{(C7)}$$

and η is introduced in place of θ because of the manner in which it enters ξ (v.i.).

Eq. (C5) has only one real root, opposite in sign to θ, if $|\eta| = |3^{1/2}\theta| < 1$, while if $|\eta| > 1$ there are three real roots; solving for these roots yields

$$\xi_m = -3^{-1/2}\eta \left\{ \left[1 + (1 - \eta^6)^{1/2}\right]^{1/3} + \left[1 - (1 - \eta^6)^{1/2}\right]^{1/3} \right\}, \quad |\eta| < 1 \qquad \text{(C8a)}$$

$$= \frac{-2}{\sqrt{3}}\eta\,|\eta| \cos \left[\frac{\phi}{3} + \begin{pmatrix} 2\pi/3 \\ 0 \\ -2\pi/3 \end{pmatrix} \right], \quad |\eta| > 1 \qquad \text{(C8b)}$$

where

$$\cos\phi = |\eta|^{-3}. \qquad \text{(C9)}$$

If $|\eta| > 1$ that value of ξ yielding the maximum value of h is to be chosen.

The limiting values of ξ for large $|\eta|$ are

$$\xi_m = \pm\eta\,|\eta| - 3^{-3/2}\eta^{-1}, \quad 2 \cdot 3^{-3/2}\eta^{-1}. \qquad \text{(C10)}$$

The corresponding, asymptotic forms of h are

$$h \sim 2^{1/3}/3\eta, \qquad \eta \gg 1 \qquad \text{(C11a)}$$

$$\sim 2^{4/3} \cdot 3^{-1/4}\,|\eta|^{1/2}, \qquad -\eta \gg 1. \qquad \text{(C11b)}$$

We note that these approximations actually are quite adequate for $|\eta| > 1$.

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